# INDEX

## AUTHORS

TITLES

PROBLEMS

*Proposals, Solutions and Quickies are indexed below by means of the code letters* P, S, *and* Q, *respectively. Page numbers are given in parentheses. Thus,* P1078(258) *refers to proposal number* 1078 *which appears on page* 258.

NEWS AND LETTERS



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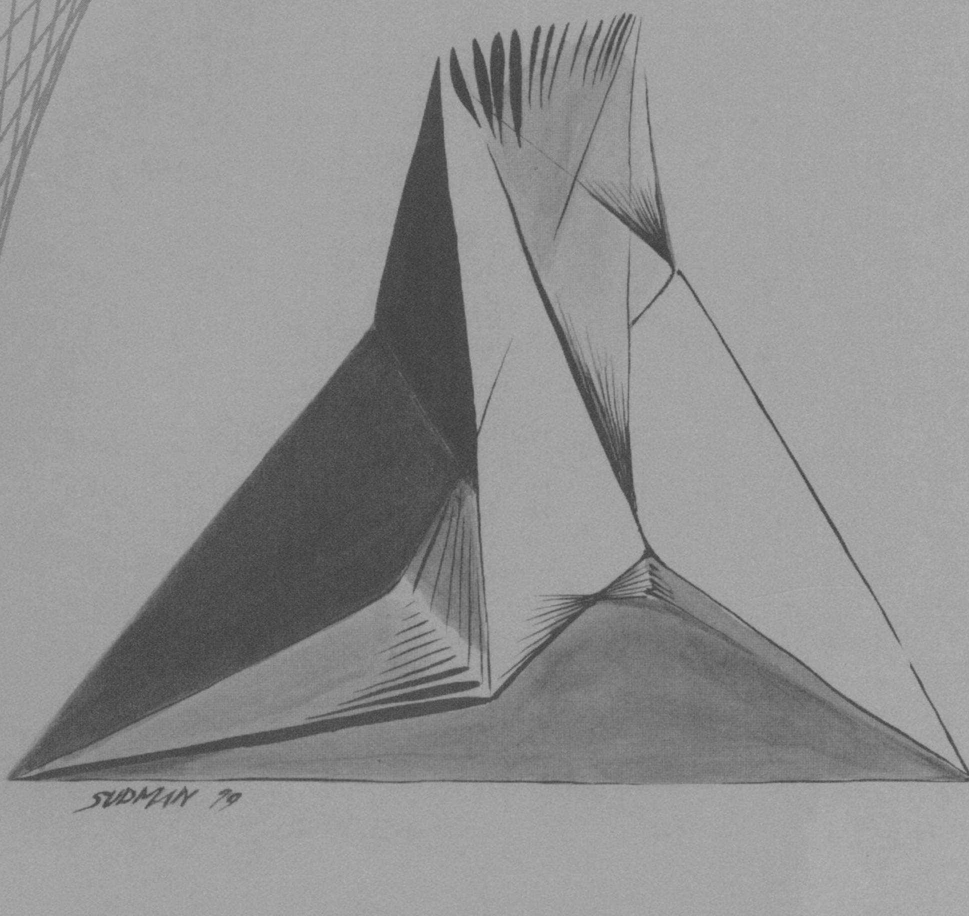

Vol. 52, No. 5
November, 1979

FLABBY POLYHEDRA • LOOPING PROGRAMS

TO OBEY OR NOT TO OBEY • WEIRD DICE

# MATHEMATICS MAGAZINE

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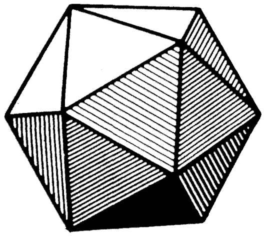



**COVER:** A simple polyhedron that flexes! For a report on this and other flexible polyhedra, see page 275.

## ARTICLES



## NOTES



## PROBLEMS



## REVIEWS



## NEWS AND LETTERS



## INDEX

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## ABOUT OUR AUTHORS

**Robert Connelly** ("The Rigidity of Polyhedral Surfaces") received his Ph.D. degree in "geometric piecewise-linear topology" at the University of Michigan in 1969. In 1973 he was seduced by the "rigidity conjecture." Dave Singer, who had an office next to his at Cornell, first showed him the problem and introduced him to Herman Gluck's very stimulating paper. After obtaining some encouraging partial results he spent a very pleasant year (1975–1976) at I.H.E.S. in France trying to prove the conjecture, with no success. Finally in June 1977, after learning that the NSF was still willing to sponsor his research, he found a counterexample. He is currently a visiting assistant professor at Cornell University.

**Arthur Charlesworth** ("Infinite Loops in Computer Programs") received his Ph.D. from Duke University in set theoretic topology, the area of most of his published research. While at Duke, he took a variety of courses in the foundations of both mathematics and computer science and during his first year at the University of Richmond, 1976–77, he presented a weekly faculty seminar on some of the remarkable discoveries in this area, such as those of Gödel and Turing. Since then he has been engaged in developing proofs of these results which are easily understandable by the general mathematician and also suitable for use in seminars for undergraduate students. The present article resulted from this work.

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# The Rigidity of Polyhedral Surfaces

*1813 Cauchy: all convex polyhedral surfaces are rigid.*
*1974 Gluck: almost all triangulated surfaces are rigid.*
*1977 Connelly: not all triangulated surfaces are rigid.*

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*Cornell University*
*Ithaca, NY 14853*

Are triangulated polyhedral surfaces rigid? Euler apparently thought so for he said, "A closed spacial figure allows no changes, as long as it is not ripped apart." However, proving such "rigidity" statements is another matter. In 1813 the famous French mathematician A. L. Cauchy proved that a convex polyhedral surface is rigid if its flat polygonal faces are held rigid. In 1896 R. Bricard, a French engineer, showed that the only flexible octahedra had bad self-intersections, and so all embedded octahedra (i.e., those that can be represented in 3-dimensional space without self-intersections) were rigid. In the 1940's A. D. Alexandrov, a Russian geometer, showed that all triangulated convex polyhedral surfaces were rigid, if there were no vertices in the interior of the flat natural faces. Then in 1974 H. Gluck showed that "almost all" triangulated spherical surfaces were rigid.

Despite all this "evidence" for the rigidity conjecture, that all embedded polyhedral surfaces are rigid, in June 1977 I found a counterexample—an embedded polyhedral surface that flexes. After a brief introduction to what is meant by rigidity, we will see how to build some models and discuss why they work. We will conclude with some related results, a few conjectures, and several open questions.

## Rigidity

In order to get a feeling for what is meant by rigidity, we present first a few definitions and examples in Euclidean $n$-space $E^n$. A **framework** $F$ in $E^3$ or $E^2$ is a collection of **vertices**, points in $E^3$ or $E^2$, together with a collection of **rods** joining certain pairs of the vertices. We think of a framework as a collection of dowel rods joined with flexible rubber connectors at their endpoints. A framework $F$ is called **rigid** if any continuous motion of the vertices that keeps the length of every rod fixed also keeps fixed the distance between every pair of vertices in the framework. For example, the frameworks of FIGURE 1 are rigid in $E^3$. The last of these frameworks, the convex octahedron, is one of the "surfaces" that Cauchy's theorem shows is

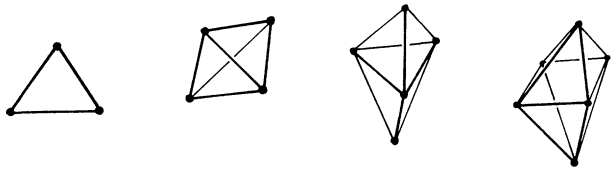


FIGURE 1

rigid. FIGURE 2 contains some examples of frameworks that are not rigid in $E^3$: each of these permit a continuous motion, or flex, of the framework that changes its shape.

Suppose we have a finite collection of triangles in $E^3$, where any two intersect in a common edge, a common vertex, or not at all, and at any vertex the triangles intersecting that vertex form a polygonal disk with the vertex in its interior. Then the union of the 2-dimensional triangles is called a **polyhedral surface**, and the collection of triangles is called a **triangulation** of that surface. For instance, the last three examples in FIGURE 1 are triangulated surfaces. The vertices and edges of these triangles form the framework associated with this triangulation of the surface. For our purposes when we talk about the rigidity or flexibility of a polyhedral surface, we will mean the rigidity or flexibility of the framework associated with some triangulation. So the rigidity conjecture says that any triangulation of a polyhedral surface in $E^3$ is rigid. This is false, and we can now start with a description of the counterexample.

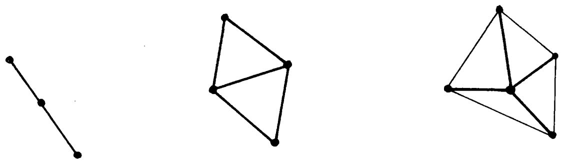


FIGURE 2

## The construction

To understand why the examples flex it is helpful to describe one of the types of flexible octahedra of Bricard. The construction depends on a little lemma from Euclidean geometry, illustrated in FIGURE 3.

LEMMA. *Let $aba'b'$ be a 4-gon in $E^3$ with equal opposite sides $ab = a'b'$, $a'b = ab'$. (Assume $a, b, a', b'$ are not all on one line.) Then there is a unique line $L$ meeting the diagonals $aa'$, $bb'$ in their centers such that rotation by $180°$ about $L$ leaves the 4-gon invariant by interchanging $a$ with $a'$ and $b$ with $b'$.*

The proof of the lemma is not hard, and is the only part of the whole construction that requires any concentration. If, on the one hand, the midpoints of the diagonals $aa'$ and $bb'$ coincide, then $aba'b'$ is planar (and in fact is a parallelogram). Then $L$ must be the line through

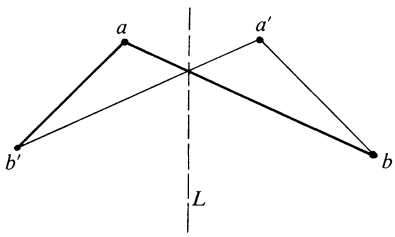



FIGURE 3

this common midpoint perpendicular to the plane of the 4-gon $aba'b'$. So suppose, on the other hand, that the midpoints $x,y$ of $aa'$ and $bb'$, respectively, are distinct. Then all we have to show is that the line $L = xy$ is perpendicular to $aa'$ and $bb'$, for clearly a 180° rotation about such a line will interchange $a$ with $a'$, $b$ with $b'$. But $abb'$ is congruent to $a'b'b$ by the conditions on the lengths of the sides. So the median lengths $a'y$ and $ay$ are equal (see FIGURE 4). So $yaa'$ is an isosceles triangle and $xy$ is the median to the base. Thus, $L = xy$ is perpendicular to $aa'$. Similarly $L$ is perpendicular to $bb'$. This completes the proof.

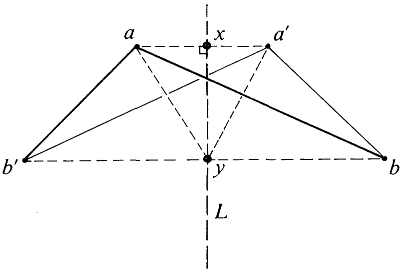



FIGURE 4

With this lemma in mind we can construct some of the octahedra discovered by Bricard and show why they are flexible. We regard an octahedron as a framework with a 4-gon $aba'b'$ and 2 other vertices $c,c'$. In addition to the rods on the 4-gon $aba'b'$, $c$ and $c'$ each have four rods connecting them to each vertex on the 4-gon.

Start with a 4-gon $aba'b'$ (as in the lemma) with opposite sides equal. Choose $c$ anywhere not on $L$, its line of symmetry. With $c$ joined to the 4-gon $aba'b'$ we get a framework something like the last framework in FIGURE 2. If $aba'b'$ is not coplanar or if $aba'b'$ is a parallelogram and $c$ is inside the parallelogram, this framework with 5 vertices $c(aba'b')$ will flex in $E^3$. So flex it and join it at each instant to the congruent framework $c'(a'b'ab)$ obtained from the first by rotation by 180° about $L$. The union is one of the flexible octahedra of Bricard and is easy to build with straws and strings. It is illustrated in FIGURE 5, where $L$ is vertical and the 4-gon $aba'b'$ is in the plane of the paper as a rectangle. Points $c$ and thus $c'$ are chosen inside the rectangle as shown. As this framework is flexed the vertices do not remain coplanar, but this is a very convenient position in which to start.

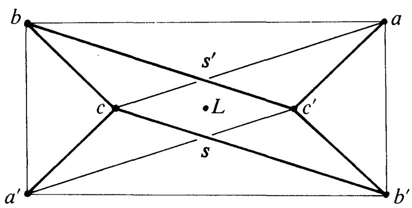



FIGURE 5

Another slight variation on this framework is to start with the points $a$ and $a'$ in a horizontal plane $H$ as in FIGURE 5. Then choose points $b,b'$ at a height $\varepsilon > 0$ above $H$ and $c,c'$ at a height $\delta > \varepsilon$ above $H$ so that all the points project orthogonally onto the picture of FIGURE 5. Line $L$ is again perpendicular to $H$, the octahedral framework is still flexible, and the boundaries of the triangles $ab'c$ and $a'bc'$ link, i.e., they cannot be pulled apart without intersecting.

To construct the embedded flexible surface we start with the surface that is used for FIGURE 5. Instead of filling in all the triangles with flat planar pieces we change the surface somewhat, but still keep the rods of the old surface. We regard the octahedral surface as being made of two pieces, a bottom and a top. Let us say the bottom is as in FIGURE 6. We push down on each of

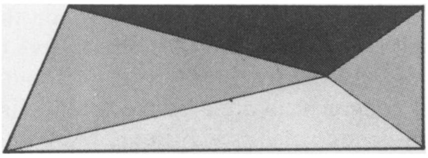


FIGURE 6

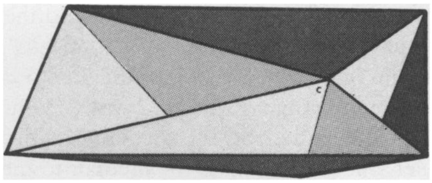


FIGURE 7

the triangular faces to get a new surface that looks like FIGURE 7, in which each triangle is replaced with an upsidedown bottomless tetrahedron, a pit.

Similarly we replace the "top" surface, FIGURE 8, with the surface in FIGURE 9, where each triangle is replaced by a bottomless pyramid, a mountain. Just as the surfaces of FIGURES 6 and 8 flex, so do the surfaces of FIGURES 7 and 9, with the extra vertices, the apexes of the pyramids, moving rigidly with respect to their bases.

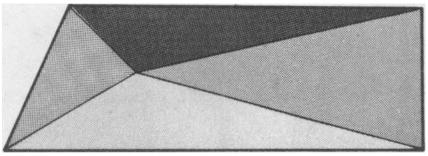


FIGURE 8

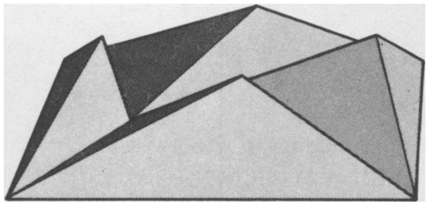


FIGURE 9

We next glue the surfaces of FIGURES 7 and 9 together along their common boundary to get the surface of FIGURE 10. This surface is flexible, just like the surface of FIGURE 5, but unfortunately it has a couple of self-interactions $s$ and $s'$. FIGURE 11 shows the parts of the surface of FIGURE 10 that intersect: points $s$ and $s'$ correspond to the crossing points of FIGURE 5.

In order to get rid of $s$ and $s'$, we build what I call a crinkle, which is again based on the Bricard flexible octahedra. Choose a planar 4-gon $defg$ with opposite sides equal, $de=fg, ef=gd$ as in FIGURE 12 with the segment $de$ intersecting $fg$. Choose a point $h$ directly over the center of the circle through $defg$ and $h'$ the same distance under the center. Thus $hd=he=hf=hg=h'd=h'e=h'f=h'g$. Then the frameworks $h(defg)$ and $h'(defg)$ flex in conjunction. (The 4-gon $defg$ actually remains coplanar.) The union of the triangular faces $hef, hfg, hgd, h'ef, h'fg, h'gd$ is the

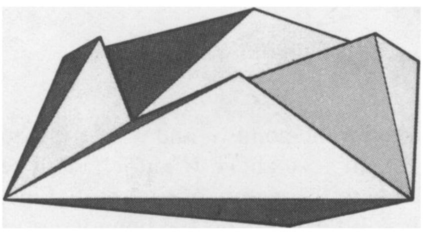


FIGURE 10

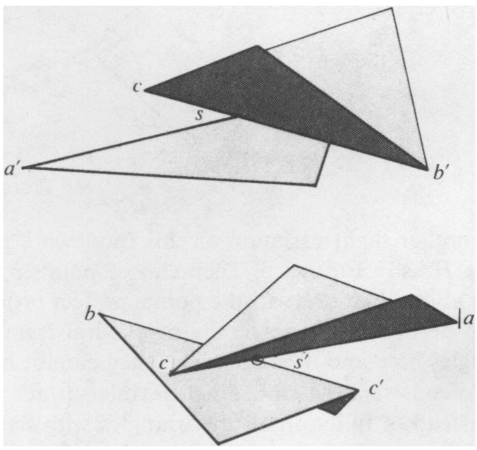



FIGURE 11

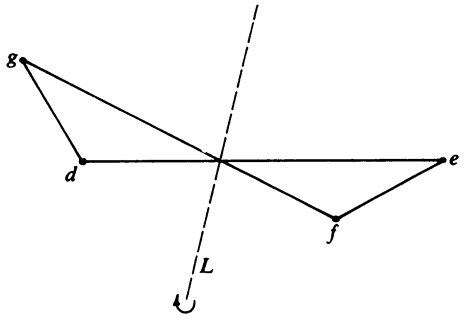



FIGURE 12

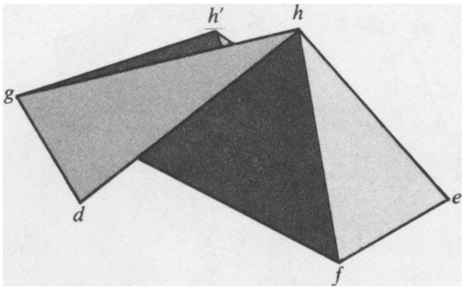



FIGURE 13

crinkle, an octahedron with two triangular faces removed (see FIGURE 13), with boundary *hdh′e*. The distance from *d* to *e* remains fixed during the flex, since it would still flex if the *de* rod were there.

To construct the final embedded flexible surface, take the surface of FIGURE 10 and cut out, as in FIGURE 14, one small quadrilateral hole around each of the self-intersection points. Then insert a crinkle of the appropriate size into each of the holes. If the crinkle is positioned properly, there will be no self-intersections in the resulting surface (FIGURE 15). Since *de* remains at a fixed distance in the crinkle, the surface with the two holes removed and the crinkle flex in conjunction. Thus the whole crinkled surface flexes. It looks something like FIGURE 16.

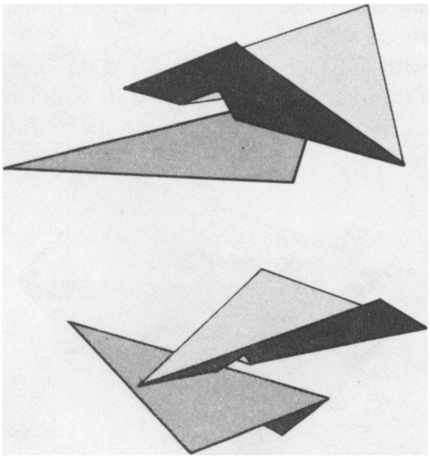


FIGURE 14

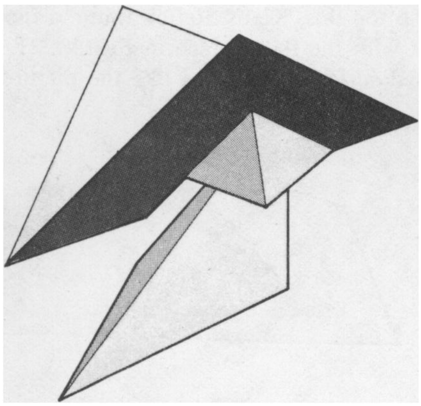


FIGURE 15

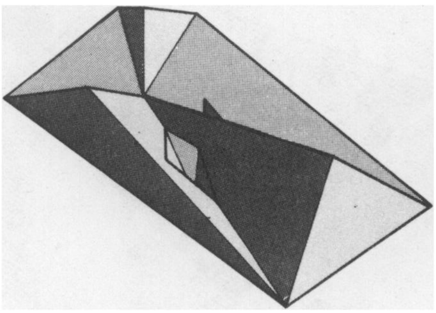


FIGURE 16

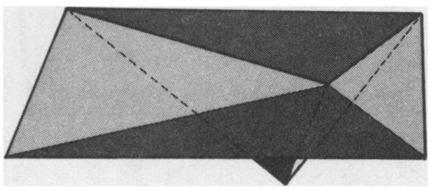


FIGURE 17

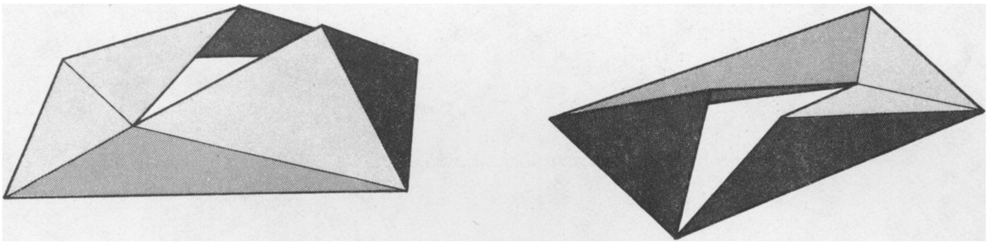


FIGURE 18

This surface was one of the first I found, and I paid no attention to the simplicity of construction or how few vertices would be needed. Subsequently N. H. Kuiper and Pierre Deligne modified my construction to get a surface with 11 vertices and 18 faces. They start with the framework described just after FIGURE 5. Instead of adding four pits to the bottom surface, they add only one as in FIGURE 17. The other three triangles are kept flat. For the upper surface they only add two mountains as in FIGURE 18 (shown with two views with two sides removed).

When these new upper and lower surfaces are glued together along their common boundary, the line segment $c'b$ intersects the sides of the two mountains above $ca$. Due to the slight raising of $c, c', b, b'$ this is the only place where the surface intersects itself. They then remove $ca$ and the inside of the two triangles with $ca$ as an edge and place a carefully proportioned crinkle, as in FIGURE 13, in the hole that was created. Points $d$ and $e$ in FIGURE 13 fit into $c$ and $a$ respectively, and the apexes of the two mountains are $h$ and $h'$. FIGURE 19 shows two views of the upper and lower surfaces glued together with $ca$ removed from the upper surface. FIGURE 20 shows two views of the final flexible surface with the crinkle added.

To top this, Klaus Steffen found a flexible surface with only 9 vertices. To build this surface, start with the flat surface in FIGURE 21. If it is cut out and folded as indicated, it will form a closed surface something like the picture in FIGURE 22. Note that in FIGURE 21 the surface is

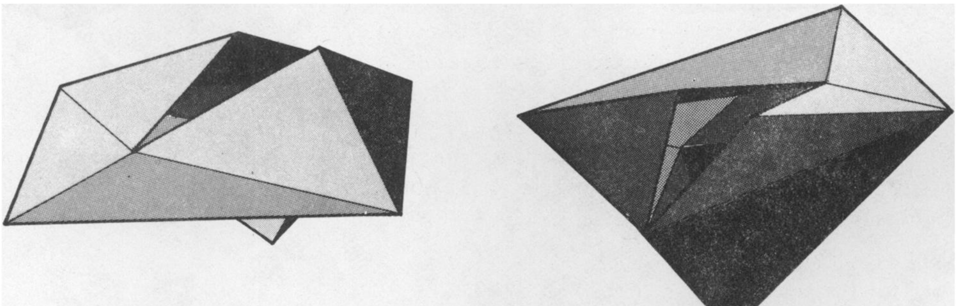


FIGURE 19

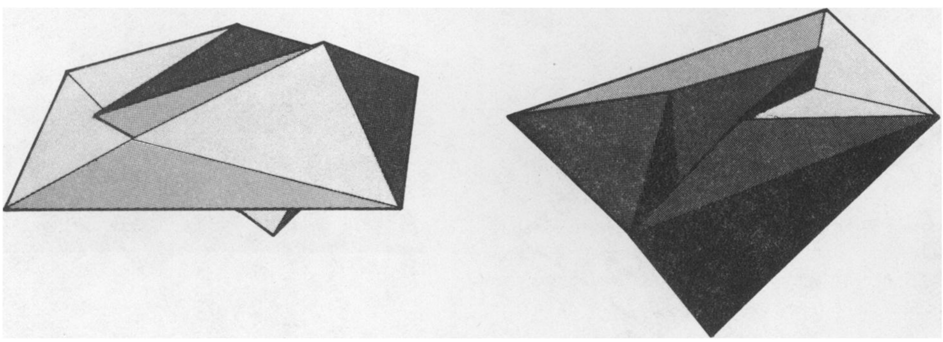


FIGURE 20

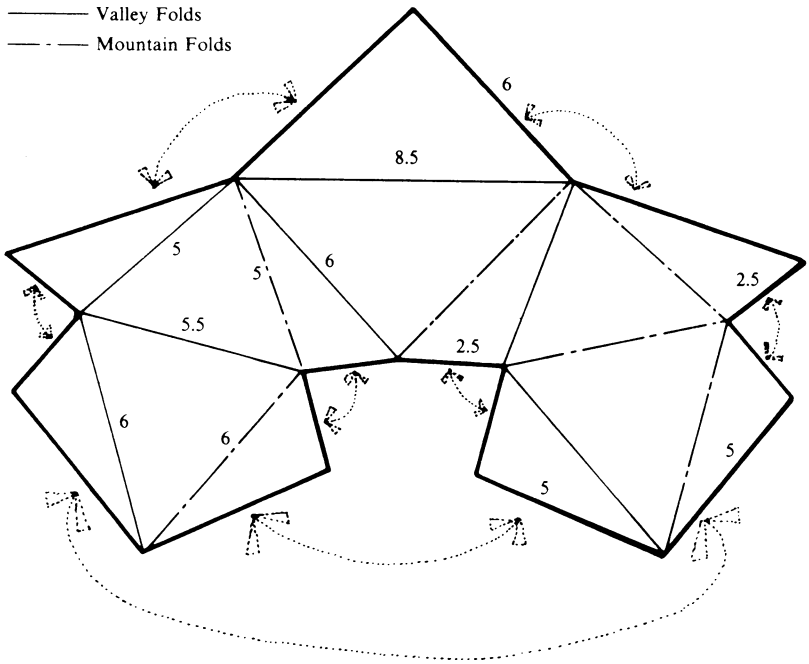


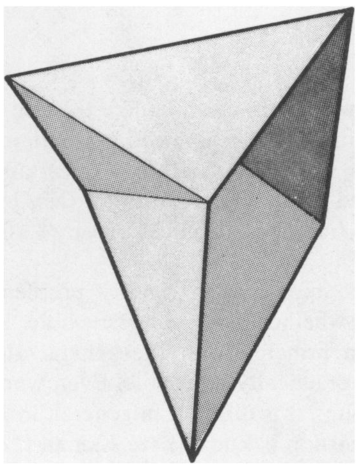

FIGURE 21

symmetric about a horizontal line and the numbers indicate appropriate lengths. If a line is not labeled, then its length can be found by looking at its symmetric length above or below, which is labeled. The two large central triangles separate the surface into two crinkles. To flex the surface hold the top two triangles in one hand and move the bottom vertex left and right.

The first model is what I describe in [**9**]. The second model is described in [**10**] and [**21**]. A nice description of the first model and related results can be found in [**18**].

## Some conjectures

An interesting property of the previous examples is that as they flex, the volume enclosed by these surfaces remains constant. However, I do not know how to prove that the volume is constant for *every* possible flexible surface.



**Cut and fold Figure 21 to obtain this flexible polyhedron with only nine vertices. This surface, found by Klaus Steffen, represents the simplest known flexible polyhedron in 3-space. Is it the best possible?**

FIGURE 22

CONJECTURE 1: *If a triangulated polyhedral surface flexes, its volume remains constant during the flex.*

Even more startling things seem to be true. Let $P$ and $P'$ be two 3-dimensional polyhedra in 3-space. We say $P$ is equivalent to $P'$ by dissection, and write $P \sim P'$, if we can dissect $P$ into a finite number of polyhedral pieces, $P_1, P_2, \ldots, P_k$ and then reassemble them to get $P'$. (So $P = P_1 \cup \cdots \cup P_k$ $P_i \cap P_j \subset$(boundary $P_i$)$\cup$(boundary $P_j$), for $i \neq j$, $P' = P_1' \cup \cdots \cup P_k'$, $P_i' \cap P_j' \subset$ (boundary $P_i'$)$\cup$(boundary $P_j'$), for $i \neq j$, and $P_i$ is congruent to $P_i'$ for $i = 1, \ldots, k$.) It turns out that a result of M. Dehn [12] answering Hilbert's third problem [14] shows that the regular cube and tetrahedron of the same volume are *not* equivalent by dissection. However, suppose that $P_t$ is the 3-dimensional solid enclosed by one of the above flexible surfaces at time $t$. A result of Sydler [27] implies that $P_0 \sim P_t$ for all $t$ in the flexing interval. A good discussion of Hilbert's third problem and this (non-trivial) result of Sydler can be found in the book of Boltianskii [4]. Still the general question remains.

CONJECTURE 2: *If $P_t$ is the polyhedral solid enclosed at time $t$ by any flexing polyhedral surface, then $P_0 \sim P_t$ for all $t$.*

Even to see explicitly the dissections for the surfaces described above would be interesting.

## Other results and open questions

The rigidity conjecture was an attempt to show that there was a more general setting for rigidity than convexity. However, since the conjecture is false, perhaps convex surfaces are natural objects to study after all. Yet despite Cauchy's result [6] and work of Alexandrov [1], it was not known until recently, Connelly [11], that an arbitrarily triangulated convex polyhedral surface (for example, the triangulated cube of FIGURE 23) was rigid.

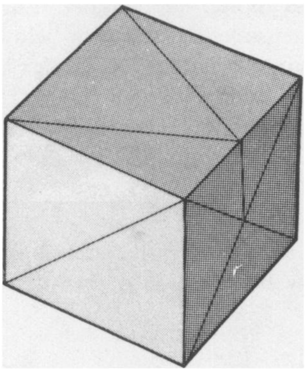


FIGURE 23

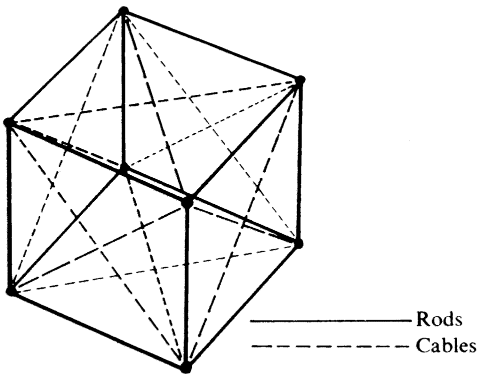



FIGURE 24

To show that frameworks like the one in FIGURE 23 are rigid, it is often helpful to show first that certain cabled frameworks are rigid (see B. Grünbaum [16]). A cabled framework is a framework with rods together with certain pairs of vertices, connected by tightened cables, that are allowed to get closer during a flex, but are not allowed to lengthen. FIGURE 24 shows one such rigid framework conjectured by Grünbaum to be rigid and proved to be rigid in [11] and [28].

Another approach to rigidity problems is to define some specific framework and try to determine whether it is rigid or flexible. Recent work of P. Kahn [17] shows that this is always possible in principle, but the general algorithm takes an enormously long number of steps, making it practically infeasible. Even when you are allowed to change the lengths of the rods a small amount, it is difficult in general, in $E^3$, to tell if it is possible to find a rigid framework. (In $E^2$ this question is known; see Laman [22] or Asimow and Roth [3].)

In the category of smooth surfaces the situation is similar but more difficult. The analogue of Cauchy's theorem, that a smooth convex surface is rigid, was first proved by Cohn-Vossen [**8**], and now there is a very short proof by Herglotz [**13**]. (See also Stoker [**26**, p. 365], Chern [**7**], or Pogorelov [**24**], for example.) The methods of my counterexample do not apply in the smooth category. However, if by smooth we mean that the map that defines the embedding of the smooth manifold is just $C^1$ (that is, continuously differentiable), then methods of Kuiper [**18**], [**19**] following Nash [**23**] provide counterexamples even for the standard, but strangely embedded, round 2-sphere. In the class of $C^2$ or $C^\infty$ embeddings, the rigidity conjecture remains open.

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# Infinite Loops in Computer Programs

*Why can't computer systems reject programs which have infinite loops?*


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It is not unusual for students in an introductory Basic programming course to write programs containing branching errors as in the following examples.

SUMNUM

```
10 REM: THIS PROGRAM FINDS THE
20 REM: SUM OF THE FIRST 1000
30 REM: NATURAL NUMBERS.
40 LET S=N=0
50 LET N=N+1
60 IF N>1000 THEN 90
70 LET S=S+N
80 GO TO 40
90 PRINT "THE SUM IS"; S
100 END
```

PRIMES

```
10 REM: THIS PROGRAM FINDS THE
20 REM: PRIMES BETWEEN 4 AND 1000.
30 FOR N=4 TO 1000
40 FOR I=2 TO N-1
50 REM: DOES I DIVIDE N?
60 IF N/I=INT(N/I) THEN 30
70 NEXT I
80 PRINT N
90 NEXT N
100 END
```

When run, each of these Basic programs goes into what is commonly called an "infinite loop": the procedure described by the program will never terminate and thus the program will continue to execute until a prearranged time limit has been reached or until someone decides to intervene. After finding that printed answers do not appear at their interactive terminal within a reasonable length of time, students would, one hopes, take a closer look and see that line 80 in SUMNUM should be changed to GO TO 50 and the branch in line 60 of PRIMES should be to line 90 instead of line 30. (Notice that if 4 is replaced by 5 in line 30 of PRIMES, the new program goes into a type of infinite loop which executes a print statement over and over, producing a list of 5's.)

Students who look ahead are apt to wonder what to do with longer and more complicated programs. Such programs might take several minutes, perhaps hours, to print out their results; if a long pause occurs at the interactive terminal after the program has begun executing, how can one be sure the program is not in an infinite loop? Computers routinely tell us of many types of errors in programs. (For example, most systems which execute Basic will refuse to run SUMNUM if line 90 is omitted, printing a message such as UNDEFINED STATEMENT REFERENCE IN LINE 60.) It is thus natural to wonder why no one had programmed the computer to check for infinite loops in programs before running them.

## Turing's Theorem

In a classic article published over forty years ago, A. M. Turing has argued that *no computer could ever be programmed to solve the problem of detecting infinite loops in computer programs*. In this paper we present a nontechnical proof of Turing's Theorem, accessible to those familiar with the Basic language.

FERMAT

```
10 REM: THIS PROGRAM PRINTS A COUNTEREXAMPLE TO A CONJECTURE
20 REM: OF FERMAT. (IF THERE IS A COUNTEREXAMPLE.)
30 LET I=3
40 FOR N=3 TO I
50 FOR X=1 TO I
60 FOR Y=1 TO I
70 FOR Z=1 TO I
80 IF X↑N+Y↑N=Z↑N THEN 150
90 NEXT Z
100 NEXT Y
110 NEXT X
120 NEXT N
130 LET I=I+1
140 GO TO 40
150 PRINT "X= ";X;" Y=";Y;" Z=";Z;" N=";N
160 END
```

Suppose the above Basic program is run on an ideal computer capable of storing and performing arithmetic on numbers of unlimited size. Will the program go into an infinite loop? The answer, of course, depends upon the unknown resolution of the celebrated conjecture of Fermat: there do not exist positive integers $x, y, z$, and $n$ (with $n>2$) such that $x^n+y^n=z^n$. Turing's Theorem asserts that no single computer program can be written which could determine, for every Basic program $P$, whether or not $P$ goes into an infinite loop.

We first prove that no program can be written *in Basic* to detect those Basic programs which would go into infinite loops. We want to make it clear that this is not the case simply because of some unique limitation of the Basic language; thus we shall freely assume an extended version of Basic. In particular, we shall place no upper bound on the number of statements in a program and shall assume that our Basic is being run on a computer having unlimited memory. In addition we shall use the symbol % to enclose data items which are strings and such strings shall be permitted to contain up to 72 characters. (The reason for these last two assumptions will soon be clear.)

It is easy to see that whether or not a program will go into an infinite loop can depend upon the data, if any, which is to be read by the program. For example, consider the following program.

```
      SAMPLE
10 READ N
20 IF N≠1 THEN 50
30 GO TO 40
40 GO TO 30
50 PRINT N
60 STOP
       (data)
100 END
```

This program goes into an infinite loop if and only if the first element of data is the number 1. In Basic, a single letter, such as N, represents a number variable; we assume a computer system such that if the first element of data read by SAMPLE were not a number but a string, such as "SMITH", a fatal error would be printed out and the program would stop. Since we will certainly not want the computer stopping a program just because it has used too much time, we will assume our programs will be executed until a STOP or END statement is reached or until a fatal error occurs.

A few more examples of the effect of data on a program will help prepare us for the proof of Turing's Theorem. First consider

```
                    BRANCH
1000 DIM A$(72)
1010 REM: THIS PROGRAM READS BASIC STATEMENTS AND
1020 REM: PRINTS ALL UNCONDITIONAL BRANCH STATEMENTS.
1030 READ A$
1040 IF A$(6,10)="GO TO" THEN 1070
1050 IF A$(6,9)="STOP" THEN 1090
1060 GO TO 1030
1070 PRINT A$
1080 GO TO 1030
1090 STOP
      (data)
2000 END
```

In this program, A$ is a string variable of length at most 72 characters and A$(6,10) denotes the substring of A$ consisting of the characters at positions 6 through 10.

To run properly, the Basic statements which BRANCH reads should have line numbers consisting of four digits and the last item of data should contain a line number and the string "STOP"; one must also be careful to use blanks carefully in the data "statements". To see an example of appropriate data, we need look no further than BRANCH itself! More precisely, we have in mind the following data lines.

```
1100 DATA %1000 DIM A$(72)%
1110 DATA %1010 REM: THIS PROGRAM READS BASIC STATEMENTS AND%
1120 DATA %1020 REM: PRINTS ALL UNCONDITIONAL BRANCH STATEMENTS%
1130 DATA %1030 READ A$%
1140 DATA %1040 IF A$(6,10)="GO TO" THEN 1070%
1150 DATA %1050 IF A$(6,9)="STOP"  THEN 1090%
1160 DATA %1060 GO TO 1030%
1170 DATA %1070 PRINT A$%
1180 DATA %1080 GO TO 1030%
1190 DATA %1090 STOP%
```

For convenience our extended Basic uses the percentage symbol to enclose data items which are strings. This permits us to use double quotes within the data strings. Since Basic statements can consist of up to 72 characters, our extended Basic also allows data strings to use an extra line, permitting any Basic statement to be considered as a data string.

When the above combination of program and data is executed, we say that "BRANCH is being run with BRANCH as its data". Notice that BRANCH would eventually stop when run with BRANCH as its data, and before stopping it would print

```
1060 GO TO 1030
1080 GO TO 1030.
```

Let us create a program BRMATE from BRANCH by replacing line 1060 in BRANCH by 1060 GO TO 1040. Notice that BRMATE would go into an infinite loop when run with BRMATE as its data since it would examine the string "1000 DIM A$(72)" over and over again.

We could actually consider running SUMNUM with data; since SUMNUM does not contain any read statements, this would not alter the logical flow within the program. Thus SUMNUM would go into an infinite loop when run with SUMNUM as its data. A similar statement could be made about the program PRIMES. Notice, on the other hand, that SAMPLE would stop when run with SAMPLE as its data since a fatal error would immediately occur. (The string "10 READ N" is not a number.)

**Proof of Turing's Theorem for Basic programs**

We are now ready for the proof of Turing's Theorem for Basic programs. Let us suppose there did indeed exist a Basic program which checks Basic programs to see whether or not they would go into an infinite loop when run with certain data. (We show that the existence of such a program would lead to an absurdity.)

In particular, then, we may assume a program CHECK, designed so that when it is run with a program *P* as its data,

(1) If *P* would go into an infinite loop when run with *P* as its data, CHECK prints the message "INFINITE LOOP" and then stops.

(2) If *P* would eventually stop when run with *P* as its data, CHECK prints the message "NO INFINITE LOOP" and then stops.

For example, if CHECK is run with BRANCH or SAMPLE as its data, CHECK prints "NO INFINITE LOOP" and stops. On the other hand, if CHECK is run with BRMATE or SUMNUM as its data, CHECK prints "INFINITE LOOP" and stops.

Since we are assuming that CHECK exists, we are free to do anything with CHECK that can be done with any Basic program. In particular, let us modify CHECK somewhat. Wherever the program CHECK contains a statement which causes "NO INFINITE LOOP" to be printed out, say

```
3000 PRINT "NO INFINITE LOOP",
```

let us replace it with two statements of the form

```
3000 GO TO 3005
3005 GO TO 3000.
```

Call the new program CHMATE and notice that data for which CHECK would print "NO INFINITE LOOP" will cause CHMATE to go into an infinite loop.

Thus when CHMATE is run with a program *P* as its data,

(1) If *P* would go into an infinite loop when run with *P* as its data, CHMATE prints the message "INFINITE LOOP" and then stops.

(2) If *P* would eventually stop when run with *P* as its data, CHMATE goes into an infinite loop.

Of course, the role of *P* above can be replaced by BRANCH, BRMATE, or any other program. Consider what would happen if *P* is replaced by CHMATE: we learn that when CHMATE is run with CHMATE as its data,

(1) If CHMATE would go into an infinite loop when run with CHMATE as its data, CHMATE prints the message "INFINITE LOOP" and then stops.

(2) If CHMATE would eventually stop when run with CHMATE as its data, CHMATE goes into an infinite loop.

In summary, we have learned that when CHMATE is run with CHMATE as its data, it eventually stops if it goes into an infinite loop and it goes into an infinite loop if it eventually stops. This absurdity shows that the program CHMATE, and thus CHECK itself, could never exist. Therefore we have shown that "no program can be written in Basic to detect those Basic programs which would go into infinite loops".

**Proof of Turing's Theorem assuming Church's Thesis**

In order to conclude the general version of Turing's Theorem that "no computer could ever be programmed to solve the problem of detecting infinite loops in computer programs", it would clearly be sufficient to show that any algorithm which eventually terminates (and thus any

## A Definition of Recursive Functions

The mathematical notion of "recursive" function is a precise attempt to single out exactly those functions $f$ whose values can be calculated using an algorithm which terminates after a finite number of steps. There are many equivalent ways to define the recursive functions. One approach (see [7], page 109) is to define a recursive function as any function from a finite Cartesian product $\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ to the set $\mathbb{N}$ (of nonnegative integers) which can be constructed from the functions listed in (1) using the operations in (2) finitely often.

(1) (addition) $f(x,y) = x+y$

(multiplication) $f(x,y) = x \cdot y$

(comparison) $f(x,y) = \begin{cases} 0 \text{ if } x<y \\ 1 \text{ if } x \geqslant y \end{cases}$

(projections) For each $n$ such that $n \geqslant 1$ and each $i$ such that $i \leqslant n$, the function $f_{n,i}$ from $\mathbb{N}^n$ to $\mathbb{N}$ defined by

$$f_{n,i}(x_1, x_2, \ldots, x_n) = x_i.$$

(2) (composition) If $f$ is a function from $\mathbb{N}^m (m \geqslant 1)$ to $\mathbb{N}$ and $g_1, g_2, \ldots, g_m$ are functions from $\mathbb{N}^n (n \geqslant 1)$ to $\mathbb{N}$, then the composition of $f$ with $g_1, g_2, \ldots, g_m$ is the function $h$ from $\mathbb{N}^n$ to $\mathbb{N}$ defined by

$$h(x_1, x_2, \ldots, x_n) = f[g_1(x_1, \ldots, x_n), g_2(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)].$$

(minimization) If $g$ is a function from $\mathbb{N}^n (n \geqslant 2)$ to $\mathbb{N}$ such that for every $(x_1, x_2, \ldots, x_{n-1})$ in $\mathbb{N}^{n-1}$ there is an $x$ in $\mathbb{N}$ for which $g(x_1, x_2, \ldots, x_{n-1}, x) = 0$, then the minimization of $g$ is the function $h$ from $\mathbb{N}^{n-1}$ to $\mathbb{N}$ defined by

$$h(x_1, x_2, \ldots, x_{n-1}) = \min\{x \mid g(x_1, x_2, \ldots, x_{n-1}, x) = 0\}.$$

It is clear that each function in (1) would be programmable in an ideal version of Basic. (For example, a program could be written to accept the numbers $x$ and $y$ as input and give the number $x+y$ as output.) One can also observe that the composition of programmable functions is programmable and the minimization of a programmable function (which satisfies the required condition) is programmable. Thus it is intuitively clear that every recursive function is programmable.

**Figure 1**

candidate for CHECK) could be programmed in Basic. This follows from a principle known as Church's Thesis which was introduced by Alonzo Church in 1936 and which asserts that every algorithm which eventually terminates corresponds to a recursive function. (The definition of a recursive function is given in FIGURE 1; it is a mathematically precise concept which is intended to correspond to the informal notion of a function which is calculable by an algorithm.) The general version of Turing's Theorem follows from Church's Thesis since it is well known that every recursive function is programmable in languages such as Basic. There is very strong evidence for Church's Thesis and it is widely accepted today, although it has not been proven in a rigorous sense. Indeed it may be inappropriate to expect a rigorous proof that a vague concept such as algorithm actually corresponds to its intended mathematical formulation. For more about Church's Thesis, see Chapter 12 of [4].

### An alternative to Church's Thesis

We can thus extend our proof to a proof of the general Turing Theorem by relying upon Church's Thesis. Although it is not unusual to use Church's Thesis in this way, let us see how it may be replaced in this instance by a different assumption. Our wish, of course, is to conclude that if CHECK were written in any given computer programming language and CHECK could correctly process both Basic programs and programs in the given language, the absurdity obtained by constructing CHMATE would still arise. If the given programming language will not permit programs to be read as data (roughly, if the language does not satisfy analogues to the conditions of our extended Basic), then a program like CHECK could not exist, so we may assume that the language permits programs to be data. Notice that we made fundamental use of only one additional property in going from CHECK to CHMATE: that we could replace a print statement in a program by statements which cause an infinite loop to occur. Thus we may conclude that CHECK cannot be written in any programming language if we are willing to assume the following proposition: any programming language which is powerful enough in which to write a program like CHECK must also allow statements which can be combined to form an infinite loop.

This proposition would be false if it is possible to develop a programming language that is powerful and that, in addition, incorporates a clever new kind of loop structure which guarantees that all loops terminate. To demonstrate that such a programming language is very unlikely, we show the following: if $\mathfrak{L}$ is a programming language which is powerful enough to write an $\mathfrak{L}$-syntax checking program, an $\mathfrak{L}$-interpreter, and a program which has the same effect as the program CREATE in FIGURE 2, then $\mathfrak{L}$ must also allow statements which can be combined to form an infinite loop. (By an **$\mathfrak{L}$-syntax checking** program we mean a program in $\mathfrak{L}$ which checks to see whether or not a candidate for a program in $\mathfrak{L}$ really has the correct syntax, i.e.,

CREATE

```
10 DIM C[72], C$[72], S$[72]
20 PRINT "WHAT IS THE ORIGINAL STRING OF CHARACTERS";
30 INPUT C$
40 PRINT "WHAT IS THE MAXIMUM LENGTH OF THE STRINGS YOU WANT";
50 INPUT N
60 FOR K=1 TO N
70 REM: THE STRINGS GENERATED AT THIS STAGE HAVE LENGTH K
80 FOR I=1 TO K-1
90 C[I]=1
100 NEXT I
110 C[K]=0
120 I=K
130 C[I]=C[I]+1
140 IF C[I]>LEN(C$) THEN 240
150 REM: WE NOW PUT TOGETHER THE NEXT STRING S$
160 REM: THE I'TH CHARACTER IN S$ IS DENOTED S$(I, I) AND IS ASSIGNED
170 REM: THE C(I)'TH CHARACTER IN THE ORIGINAL STRING C$
180 FOR I=1 TO K
190 S$[I, I]=C$[C[I], C[I]]
200 NEXT I
210 PRINT S$
220 I=K
230 GO TO 130
240 C[I]=1
250 I=I-1
260 IF I> =1 THEN 130
270 NEXT K
280 END
```

Given a string of characters C$, CREATE generates all strings which consist of the characters in C$ and which have length at most $N$. Thus if C$ is "AB" and $N$ is 3, then CREATE generates the strings A, B, AA, AB, BA, BB, AAA, AAB, ABA, ABB, BAA, BAB, BBA, and BBB.

**Figure 2**

the correct form. An £**-interpreter** is a program in £ which decodes and executes statements in £. Such programs are, of course, routinely included in most computer systems.)

Suppose that £ does not allow statements which can be combined to form an infinite loop; we can reach a contradiction as follows. Consider listing all the symbols used in the language £, then all pairs of such symbols, all triples of such symbols, etc.; that is, list all the finite strings $S_1, S_2, S_3, \ldots$ of symbols used in £. Since each program in £ is such a finite string, this means that we can enumerate all possible programs in £; in particular, we can enumerate those programs which accept a single number as input and have as their only output either "YES" or "NO", terminating immediately after printing one of these words. Let us label these programs $P_1, P_2, P_3, \ldots$. The contradiction arises by using a Cantor diagonal argument to construct a program $P$ in £ which is of this type, yet which is different from each $P_i$. Given the number $n$ as input, $P$ generates $S_1, S_2, S_3$, etc., until it has generated the string $P_n$. [A program like $P$ can be written in £ since we can use an idea similar to that of CREATE to generate the $S_i$'s (with C$ containing all the symbols used in £), we can examine each $S_i$ as it is generated and eliminate those which do not have the correct syntax to be a program in £, and we can check to see if those $S_i$'s which are programs have the form required for being a $P_i$.] Immediately after discovering $P_n$, our program $P$ decodes and executes the statements of $P_n$. When $P$ reaches the step in which $P_n$ asks for its input, $P$ supplies the number $n$. Since $P_n$, by assumption, doesn't go into an infinite loop, eventually $P_n$ will be ready to print either "YES" or "NO". Our program $P$ determines what answer $P_n$ would print and then $P$ prints the opposite answer and stops. Clearly $P$ qualifies as one of the $P_i$'s, yet for each $n$, the programs $P$ and $P_n$ differ in the output they give with $n$ as input. Thus $P$ is not one of the $P_i$'s, a contradiction.

### What about programs which don't have data?

Since many important programs do not read in any data, you may wonder whether a program could exist which at least detects which of these "simpler" programs would go into an infinite loop when run. Let us briefly describe how our proof of Turing's Theorem may be modified to show that such a program is impossible. For each Basic program $P$ there is a Basic program $P'$ which has no READ statements and which has the same effect when run as running the program $P$ with $P$ as data. Moreover it is straightforward to construct $P'$, given $P$. (We sketch such a construction procedure in the next paragraph.) Now assume that there is a program CHECK2 which, given any program $Q$ (having no READ statements) as data, prints exactly one of the messages "INFINITE LOOP" or "NO INFINITE LOOP", depending upon whether or not $Q$ would go into an infinite loop when run. Then we could write the program CHECK as follows: given a program $P$ as data, CHECK would first construct the program $P'$ and then CHECK would apply the steps in the program CHECK2 to $P'$. Thus the existence of CHECK2 implies the existence of CHECK, so no such program CHECK2 could exist.

To see how $P'$ can be constructed from $P$, first order all characters used in Basic and put them into a single string C$. Recall that for integers $K$ which are no greater than the length of C$, the notation C$(K, K) denotes the $K$th character in C$; we can thus consider that each such $K$ is a numerical code for a character of Basic. Begin writing $P'$ by defining C$ and by assigning values to an array C(I, J) so that C(I, J) is the numerical code for the $J$th character in the $I$th statement of $P$. Next place each statement of $P$ (preserving the order of the statements) into $P'$, with the following exceptions. Do not place any READ statements into $P'$. Wherever $P$ considers the $J$th character A$(J, J) in the $I$th statement which $P$ has read in, $P'$ should consider instead the character C$(C(I, J), C(I, J)). (If $P$ contains a string variable named C$ or an array named C, other straightforward adjustments would be required.)

### What does Turing's Theorem really mean?

You may have noticed that it would be easy to write a program CHECK3 in Basic which, when run with a program $P$ as its data, would satisfy the following: if $P$ would eventually stop when run with $P$ as its data, CHECK3 prints the message "NO INFINITE LOOP" and then stops. Indeed, CHECK3 would need to do little more than perform the statements of $P$. Of course, if $P$ went into an infinite loop instead, CHECK3 would never report it.

One could also observe that a single program can be written, perhaps based upon the program BRANCH, which would alert us to certain types of infinite loops; for example, the type found in SAMPLE and, perhaps, even those found in SUMNUM and PRIMES. One can, in addition, hope that eventually a program can be written to determine whether or not the program FERMAT (in the box at the beginning of this note) has an infinite loop.

The point of Turing's Theorem seems to be that there is an infinite variety of really different infinite loops. That is, even if for each program $P$ we knew of a program $\text{CHECK}_P$ (which would tell us whether or not $P$ goes into an infinite loop when run with $P$ as data), and even if each of the $\text{CHECK}_P$'s could accurately process a large collection of programs, there would be no way to combine some of the $\text{CHECK}_P$'s into a single program which would accurately process every program.

An important philosophical question is whether computers can think, and closely related to this, to what extent the brain can be viewed as a computer (see [2] and [3]). Turing's Theorem concerning the limitations of computers can stimulate interest in such questions, as one wonders whether Turing's ingenious discovery implies an inherent limitation to human ingenuity itself.

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## Vocabulary

*Ambiguous case*—politician
*Bit*—information from the horse's mouth
*Complex variable*—woman
*Directrix*—female director
*Epsilon*—married to Delta
*Falling body*—whodunit
*Group theory*—encounter dynamics
*Half-open interval*—alarm clock
*Im grossen problem*—obesity
*Joint variation*—arthritis
*Knots*—traffic jam
*Linear operator*—directory assistance
*Measure of dispersion*—litterbug
*Necessary & sufficient*—money
*One-to-one correspondence*—love-letters
*Plane, intercept form*—hijacker
*Quantifier, universal*—income tax
*Related rates*—cops & robbers
*Significant digit*—hitchhiker
*Transcendental function*—guru
*Upper bound*—price controls
*Volume*—rock 'n' roll
*Work (ft-lb)*—overweight jogger
$X \cup Y$—togetherness
*Zero*—count-down: the end.

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# NOTES

## Mo Fiorina's Advice to Children and Other Subordinates

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In recent work ([2], [3]) Morris Fiorina, a political scientist at the California Institute of Technology, dealt with the question of constituency influence on representatives' roll call voting by contrasting the predictions of an expected utility maximizing model with those of what he refers to as a "maintaining" model. In Fiorina's maintaining model, Congressmen are assumed to set an "acceptable" level of reelection probability and to vote in such a way as to maintain their (subjectively estimated) reelection probability at this level. Fiorina uses this model to account for a number of anomalies in congressional voting behavior.

In this note we generalize Fiorina's maintainer model so as to make it applicable to any dichotomous choice where an actor is confronted with (potential) demands or pressures from sources which may conflict with his own preferences. These outside sources may make incompatible demands, may differ in their ability to reward and/or punish the actor for disobedience, and may also differ in their likelihood of administering the promised rewards (or threatened punishments). We confine ourself to a three-actor model consisting of an actor (a child) and two outside sources of (potential) demands or pressures (her parents), and give an analysis as seen from the perspective of the actor (the child). Although the discussion is presented in terms of child-parent conflict, the analysis is a perfectly general one, applicable to a wide variety of conflict situations, e.g., a worker torn between demands of union and management; a student torn between demands of teacher and of peer group; a legislature torn between pressures from competing interest groups; or a soldier torn by conflicting commands from his immediate commanding officer vs. general directives of remote superiors.

We consider six cases, based on the extent to which the actor's preferences (e.g., those of the child) and those of the sources of demands (the parents) are in conflict. These cases are summarized in the following chart:

| | Sources (Parents) are | |
|---|---|---|
| Actor (Child) | In Agreement | Split |
| Is indifferent | Case I | Case IV |
| Prefers to do as neither source (parent) wishes | Case II | ------- |
| Prefers to do as stronger source (parent) wishes | Case III | Case V |
| Prefers to do as weaker source (parent) wishes | Case III | Case VI |

For each of these six cases we consider the implications for compliance behavior of the two alternative models used by Fiorina: maintaining and expected utility maximizing. (In a longer version of this paper, available upon request from the author, we consider the implications for compliance behavior of two other strategies called maximin and minimax regretting. (See Luce and Raiffa [**4**]; Ferejohn and Fiorina [**1**].))

The expected utility maximizing model is the standard approach to problems of choice under risk (see e.g., Luce and Raiffa [**4**]). The maintaining model appears most appropriate in a context where outcomes can be readily divided into two classes, satisfactory and unsatisfactory (e.g., winning or losing an election, or getting fired vs. not getting fired). In the context of child-parent conflict this model seems appropriate for children who are not seeking the best of all possible worlds, but merely one which provides no worse level of satisfaction than that they regard as acceptable and to which they have become accustomed.

Dear Professor Fiorina:

Sometimes my parents want me to do what they tell me to. This seems unfair. What should I do?

Virginia

Dear Virginia:

There are various cases of interest. For example, your parents may or may not be in agreement. First, let us consider the times when both your parents are in agreement on what they want you to do. We shall assume that when they reward you for obedience, the reward is worth $x$, and when they punish you for disobedience, the value of the punishment is $-z$. Since your parents are only human, they don't always watch to see what you're doing. However, let us assume that they are conscientious and always administer the proper reward (or punishment) when they know it is due, but never otherwise. Let the probability of their monitoring your behavior be labeled $c$. Finally, since you felt the need to write to me, it seems probable that your parents are somewhat strict, and therefore we shall assume $z > x$. On the other hand, I don't really know that much about you, so I will give you advice based on two different kinds of strategies, maximizing and maintaining, thereby allowing you to pick the decision rule which best fits your own personality and attitudes toward risk. If you are a maximizer you will seek the strategy which maximizes your own expected payoff; if you are a maintainer, you will seek the strategy which gives you some constant (and satisfactory level) of expected satisfaction.

*Case* I. In the absence of rewards or punishments from your parents, you'd be indifferent between doing what your parents want and doing the opposite. We may represent this decision problem in the following game matrix:

| Child: | Parents: Monitor ($\Pr = c$) | Don't Monitor ($\Pr = 1-c$) |
|---|---|---|
| Obey | $x$ | 0 |
| Disobey | $-z$ | 0 |

*Maximizers*: The expected payoff of obeying is $cx$; while the expected payoff of disobeying is $-cz$. Since the former is positive and the latter is negative, as a maximizer you always obey your parents.

*Maintainers*: As a modern child free of compulsive habits, you might wish to adopt a more hang-loose behavioral rule, e.g., to obey with probability $q$, so that your expected change in payoff is zero. This maintains your status quo level of satisfaction. To follow such a rule, you seek $q$ so that

$$qcx + q(1-c)0 + (1-q)(c)(-z) + (1-q)(1-c)0 = 0,$$

from which it follows readily that $q=z/(x+z)$. In other words, the greater your reward for obeying, the less you need obey; on the other hand, the more you're punished for disobeying, the more you should obey. Hence, under these assumptions, if parents believe you to be using a maintaining rule, they should always punish and never reward. (You probably should not reveal the results of this analysis to your parents.)

*Case* II. Suppose that what you want to do and what your parents want you to do are opposite. Assume that doing what you want generates its own reward of $r$, while not doing what you want to do has value $-y$. Let us also assume that you are more displeased at doing what you don't want to do, than you are pleased doing what you do want to do, i.e., $y>r$. Let us define the strength of your parents as $x+z$ and your strength as $y+r$. Your strength may be thought of as a measure of your ability to generate internal motivations for your actions, and thus it can be taken as a measure of ego-strength. The strength of your parents is a measure of their ability to provide external motivations for your behavior. I know, Virginia, that this may be painful for you to accept, but your parents can do more to affect your life (either positively or negatively) than you can do for yourself; hence, we shall assume $z>y$ and $x>r$. This decision problem may be represented as follows:

| Child: | Parents: Monitor ($\Pr=c$) | Don't Monitor ($\Pr=1-c$) |
|---|---|---|
| Obey | $x-y$ | $-y$ |
| Disobey | $-z+r$ | $r$ |

*Maximizers*: Maximizers choose to obey their parents iff $c>(r+y)/(x+z)$. Let us define the effective strength of an actor (child, father, mother, or combination thereof) as the product of strength and vigilance. Parental vigilance is $c$, the probability of monitoring, while for children vigilance is one, so strength and effective strength are identical. In these terms the rule for maximizing can be rephrased as "obey your parents only if your effective strength is less than their effective strength." (This result can be thought of as a special case of Proposition 11 of Fiorina [2], p. 35.) For parents of maximizers, high vigilance is the price of obedience.

*Maintainers*: Maintainers obey their parents with a probability $q=(cz-r)/(cz-r+cx-y)$. In order that there be a maintaining strategy, we require (see Fiorina [2], p. 35) $cx-y$ and $cz-r$ either be both negative, or both positive. If they are negative, then the greater the parents' effective reward, the more the child needs obey them, but the greater the parents' effective punishment, the less he needs obey them. Moreover, under these conditions, the greater the self-generated reward for behavior in compliance with a parental directive, the greater the obedience to that directive (and conversely for self-generated punishments). If, however, these terms are positive, then the impact of effective punishments or rewards on a maintainer is just the opposite. In particular, if these terms are positive, then the greater the (internally generated) value of a given preferred activity (as measured by $r$), the less often the maintaining child needs to do it. Thus, under these conditions, the things a maintainer likes to do best he needn't do often.

We might also note that, regardless of which condition holds, if their child has maintaining strategy, the more parents monitor their child's behavior, the less likely he or she is to obey them —a rather paradoxical result, indeed. Furthermore, if the terms $cx-y$ and $cz-r$ are both negative, then maintainers obey their parents with a probability less than $1/2$. Thus, Virginia, if you use a maintaining rule, even though you are not as strong as your parents you may still, under some circumstances, disobey them more than half the time!

*Case* III. Suppose that you prefer to do what your parents want you to do. We may represent this decision problem by the matrix:

| Child: | Parents: Monitor (Pr $=c$) | Don't Monitor (Pr $=1-c$) |
|---|---|---|
| Obey | $x+r$ | $r$ |
| Disobey | $-z-y$ | $-y$ |

*Maximizer*: Since obedience is a dominant strategy (i.e., it is better no matter what monitoring choice is made by the parents), a maximizer clearly obeys his parents (and satisfies his own preferences at the same time).

*Maintainer*: A maintainer obeys his parents with probability $q=(cz+y)/(cz+y+cx+r)$, which is greater than $1/2$ for all $c$. The stronger punishments are (relative to rewards), the greater the obedience; the stronger rewards are (relative to punishments), the greater the disobedience. So, even though his interests and his parents are perfectly coincident, if a child is a maintainer he will still sometimes disobey.

It is easy to see that, to maximize the child's obedience, parents should set $c=1$ if $yx<zr$, and they should set $c=0$ if $yx>zr$. Thus, if $yx>zr$, for a child who is a maintainer, the best way to insure his obedience is never to check up on him.

Now, Virginia, let us consider the cases where your father and mother are split in their preferences. We shall assume that one of your parents (whose strength we shall denote $x_1+z_1$) is stronger than the other (whose strength we shall denote $x_2+z_2$), so $x_1+z_1>x_2+z_2$. For simplicity, we shall assume that your father is the stronger parent (albeit not necessarily the most vigilant). Your mother wants you to do one thing; your father would have you do the opposite. Finally, assume that your father monitors with probability $c_1$; your mother with probability $c_2$.

*Case* IV. Suppose you would be indifferent as to what to do were it not for the possibility of rewards or punishments offered by your parents. This decision problem may be represented by

| | Father Monitors | | Father Doesn't Monitor | |
|---|---|---|---|---|
| | Mother Monitors | Mother Doesn't | Mother Monitors | Mother Doesn't |
| Probability | $c_1c_2$ | $c_1(1-c_2)$ | $(1-c_1)c_2$ | $(1-c_1)(1-c_2)$ |
| Child: Obey Father | $x_1-z_2$ | $x_1$ | $-z_2$ | 0 |
| Obey Mother | $-z_1+x_2$ | $-z_1$ | $x_2$ | 0 |

This case is equivalent to the heterogeneous constituency case in Fiorina (**[2]**, **[3]**), although our notation is somewhat different.

*Maximizers*: If $c_1=c_2$, maximizers vote exclusively with the stronger parent. For $c_1\neq c_2$, maximizers vote with the parent with greater effective strength: with the weaker parent if $c_1(z_1+x_1)<c_2(z_2+x_2)$, and with the stronger parent otherwise.

*Maintainers*: Where $c_1=c_2$, Fiorina (**[2]**, pp. 32–39) shows that for a maintaining strategy to exist requires that $x_1\geqslant z_2$; in this case the maintainer obeys the stronger parent with probability

$$q=\frac{z_1-x_2}{(x_1-z_2)+(z_1-x_2)}>\frac{1}{2}.$$

If $c_1=c_2$ and $x_1=z_2$, then the maintaining child always obeys the stronger parent. Further, if we assume (not unreasonably) that, when parents are grossly unequal in strength, $x_1\gg z_2$ ($x_1$ considerably greater than $z_2$) and $z_1\gg x_2$, then we obtain the paradoxical result that maintaining children exhibit greater variance in their choice of which parent to obey when one parent is considerably stronger than the other than in the case where both parents are of near equal strength.

Fiorina ([2], pp. 35–37) shows that if $c_1 \neq c_2$, the probability of a maintainer obeying the stronger parent is

$$q = \frac{c_1 z_1 - c_2 x_2}{c_1(x_1 + z_1) - c_2(x_2 + z_2)},$$

while maintaining strategy exists if and only if either $c_1/c_2 \leqslant x_2/z_1$, or $c_1/c_2 \geqslant z_2/x_1$. (These conditions are mutually exclusive.)

In the former case, maintainers obey the weaker parent with probability greater than 1/2 while in the latter case they obey the stronger parent with probability greater than 1/2. Hence, maintainers (like maximizers) may vote with the weaker parent provided that parent is the more vigilant. However, if $c_1 > c_2$ and $c_1/c_2 \geqslant z_2/x_1$, then the higher $c_1$ the lower the probability that a maintainer obeys the stronger parent. In other words, the more the stronger parent monitors the child's behavior, the less likely he is to obey. This result contrasts with what is true for maximizers, for whom it is always true that the greater a parent's vigilance, the more likely he or she is to be obeyed.

*Case* V. Suppose you prefer to do what the stronger parent wants you to do:

| | | Father Monitors | | Father Doesn't Monitor | |
|---|---|---|---|---|---|
| | | Mother Monitors | Mother Doesn't | Mother Monitors | Mother Doesn't |
| | Probability | $c_1c_2$ | $c_1(1-c_2)$ | $(1-c_1)c_2$ | $(1-c_1)(1-c_2)$ |
| Child: | Obey Father | $x_1 - z_2 + r$ | $x_1 + r$ | $-z_2 + r$ | $r$ |
| | Obey Mother | $-z_1 + x_2 - y$ | $-z_1 - y$ | $x_2 - y$ | $-y$ |

*Maximizers*: Maximizers obey the stronger parent when $c_1(x_1 + z_1) > c_2(x_2 + z_2) - (r + y)$. The child will disobey the stronger parent only when the expected strength of this coalition (child + stronger parent) fails to exceed the effective strength of the weaker parent. If $r + y > x_2 + z_2$, the above condition will always be true, under our assumptions. Thus, if the child is stronger than his weaker parent, he will always do what he (and his stronger parent) want him to do. For maximizing children, high vigilance can compensate (within limits) for low strength to insure obedience to the will of the weaker parent.

*Maintainers*: Maintainers must obey the stronger parent with a probability

$$q = \frac{c_1 z_1 - c_2 x_2 + y}{(c_1 z_1 - c_2 x_2 + y) + (c_1 x_1 - c_2 z_2 + r)}.$$

A maintaining strategy exists if and only if either $c_1 z_1 < c_2 x_2 - y$ or $c_1 x_1 > c_2 z_2 - r$. (These conditions are mutually exclusive.) If the former holds, then maintainers obey the weaker parent with probability greater than 1/2, while if the latter holds, then they obey the stronger parent with probability greater than 1/2. Hence, in the first case, we again obtain the paradoxical result that the higher $c_1$, the lower the probability that a maintainer obeys the stronger parent; in other words, the more the stronger parent monitors the maintaining child's behavior, the less likely is the child to obey him.

*Case* VI. Suppose your preferences coincide with those of the weaker parent. Since the analysis is straightforward, we leave this case for you to work out. As a helpful hint, if you need one, here is the decision matrix:

| | | Father Monitors | | Father Doesn't Monitor | |
|---|---|---|---|---|---|
| | | Mother Monitors | Mother Doesn't | Mother Monitors | Mother Doesn't |
| | Probability | $c_1c_2$ | $c_1(1-c_2)$ | $(1-c_1)c_2$ | $(1-c_1)(1-c_2)$ |
| Child: | Obey Father | $x_1 - z_2 - y$ | $x_1 - y$ | $-z_2 - y$ | $-y$ |
| | Obey Mother | $-z_1 + x_2 + r$ | $-z_1 + r$ | $x_2 + r$ | $r$ |

Our analysis has been written from the child's perspective. We showed how choice can be affected by the structure of the rewards (or punishments) administered externally by authority figures such as parents and also by the child's internally generated reward structure. Choices were determined as a function of these rewards. Our analysis was determined by just two decision rules which we assumed the child could adopt to govern his behavior: maximizing and maintaining.

We could extend our result in one of two ways. One natural extension would be to reexamine our decision problems from the parental perspective. We would make reasonable assumptions about the costs involved in monitoring the child's behavior and the payoff to parents of achieving obedience and determine optimal parental strategies in a 2-person or 3-person game. We might (a la Ferejohn and Fiorina [1]) construct a decision problem in which the alternatives open to parents are "to command $x$," "to command not $x$," or "to leave the child to do as he pleases." Another direction would be to look at other decision rules, e.g., maximin or minimax regret.

Our analysis led to some paradoxical conclusions (e.g., under some circumstances the less surveillance, the more compliance) and often led to non-obvious results about the differing impacts of rewards and punishments under the two different decision rules. Although presented in the context of child-parent conflict, our analysis is intended to be applicable to a wide variety of situations in which some command and others choose whether or not to obey. We leave to the reader the task of detailing such alternative scenarios.

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# Sequentially So

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Which elements in the sequence $N = \{1,2,3,4,\ldots,n,\ldots\}$ of natural numbers can be written as a sum of two or more consecutive elements of $N$? We exemplify by this simple number theoretic problem the paradigm: "Discovery consists of seeing what all have seen and thinking what nobody else has thought." Such is the process of mathematics as done by a mathematician.

One of the first things a mathematician does with a problem such as this is to look at some simple cases. So we consider the cases given in TABLE 1; note that we don't have anything to say about 2, 4, 8 or 16. From the Table it is reasonable to conjecture that odd numbers can be written as the sum of two consecutive elements of $N$, and, with a little experimentation, that a power of 2 cannot be written as a sum of consecutive elements of $N$. Also, we can see by examination that there is not always a unique answer, e.g., $9=4+5=2+3+4$.

| | |
|---|---|
| $3=1+2$ | $13=6+7$ |
| $5=2+3$ | $14=2+3+4+5$ |
| $6=1+2+3$ | $15=1+2+3+4+5=7+8$ |
| $7=3+4$ | $17=8+9$ |
| $9=4+5=2+3+4$ | $18=5+6+7$ |
| $10=1+2+3+4$ | $19=9+10$ |
| $11=5+6$ | $20=2+3+4+5+6$ |
| $12=3+4+5$ | $21=1+2+3+4+5+6=10+11$ |

TABLE 1

Since the sum of an even number and an odd number is odd, we proceed to prove our point. It is easy to see that an odd element of $N$ can be written as the sum of two consecutive elements of $N$, for if $m \in N$ is odd, it is the sum of the consecutive integers $(m-1)/2$ and $(m+1)/2$.

Regarding powers of 2, we can confirm our conjecture by proving that $2^p$ is never the sum of consecutive positive integers. To do this we note that $\Sigma_{j=1}^{n} j=[n(n+1)]/2$, so we have

$$Q=\sum_{j=0}^{n} k+j=(n+1)(k)+\frac{(n)(n+1)}{2}=\frac{(n+1)(2k+n)}{2}.$$

Hence, if $n$ is even, $n+1$ is odd and $Q$ has an odd factor, while if $n$ is odd, $2k+n$ is odd and $Q$ again has an odd factor. Thus $Q$ cannot be a power of 2.

So we now know that odd elements *can* be written as the sum of just two consecutive elements of $N$, and that powers of 2 *cannot* be written as a sum of two or more consecutive elements of $N$. What about an even element which is not a power of 2? Let's examine a few examples to seek a procedure that will work in this case.

If we consider 884 and factor 884 as $4\times 221$, we can solve our problem if we can find 4 appropriate sums for 221. We know how to write 221 as a sum of two consecutive elements of $N$, namely $221=110+111$. Now, three other "nice" sums for 221 are $109+112$, $108+113$ and $107+114$ so we have a sequence of 107, 108, 109, 110, 111, 112, 113, 114 whose sum is 884 as desired. Similarly, $1470=6\times 245$, $245=122+123$, and we can generate the sequence $117, 118, 119, \ldots, 126, 127, 128$ whose sum is 1470. But, considering 710, we observe $710=5\times 142$, and so a solution can be found if we find 5 sums of 142. As above, this can be obtained by the sequence 140, 141, 142, 143, 144, noting $140+144=2(142)=141+143$. Similarly, if we consider $1484=7\times 212$, we obtain the sequence 209, 210, 211, 212, 213, 214, 215 whose sum if 1484.

With these examples in hand, we are ready to show how to write all even elements which are not powers of 2 as sums of consecutive elements in $N$. If $m$ is even but not a power of 2, then $m=r\times s$ where $r$ is odd ($r\geqslant 3$) and $s$ is even. If on the one hand $(r+1)/2>s$, we consider the two consecutive elements in $N$ whose sum is $r$, namely $(r-1)/2$ and $(r+1)/2$. Working "both directions" from $(r-1)/2$ and $(r+1)/2$, we obtain a sequence of positive integers of length $2s$. On the other hand, if $(r+1)/2\leqslant s$ we look at $s$ and work "both directions" from $s$ to obtain a sequence of length $r$.

We leave unanswered some obvious questions concerning uniqueness. We also note that similar results are obtained when we consider sequences of multiples of natural numbers, e.g., $K=\{5,10,15,20,\ldots\}$. For instance, an element of $P=\{5,10,20,40,80,160,\ldots\}$, cannot be written as a sum of consecutive elements of $K$. Proofs of such theorems are similar to the ones presented above and provide extensions of our first result.

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# The Utilities Problem

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An old, yet ever popular recreational problem is the utilities problem, also known as the water, gas, and electricity problem, which is usually stated as follows:

> Try to install water, gas, and electrical lines from utilities $W$, $G$, and $E$ to each of the houses $A$, $B$, and $C$ without any line crossing another (FIGURE 1).

A number of other problems are equivalent to this one. For example, if we replace the utilities by three wells, $x,y$, and $z$, we get the bad neighbors problem or the houses and wells problem [**8**, p. 14]:

> Three persons live on adjacent lots, each provided with a well. Occasionally one or more of the wells run dry, so each person requires a path to each well. After a while the residents develop strong dislikes for each other and decide to construct their paths in such a way that they will avoid meeting each other on their way to and from the wells. Is such an arrangement possible?

A more violent version is the Corsican vendetta [**2**, pp. 80–81]:

> There are three families such that any member of one family will attempt to kill any member of another family whenever their paths cross. However, the well, the market, and the church are, by tradition, neutral places which are free from violence. Therefore, the families would like to take paths from their homes to the three safe places so that the paths of different families will never cross.

Any problem with this many variations must have roots which go way back in history. In fact, most published references to the problem characterize it as "very ancient." However, the earliest published reference to the problem known to this author dates only as far back as 1917. In that year, the English puzzlist H. E. Dudeney included the utility problem in his book *Amusements in Mathematics* [**3**, Problem 251]. Dudeney himself characterizes the problem as "old as the hills,...much older than electric lighting, or even gas," yet he gives no earlier sources. Sam Loyd, Jr., claimed that his father, an American contemporary of Dudeney, "brought out" the utility problem in 1900 [**1**, p. 142]. Loyd does not claim that his father invented the puzzle,

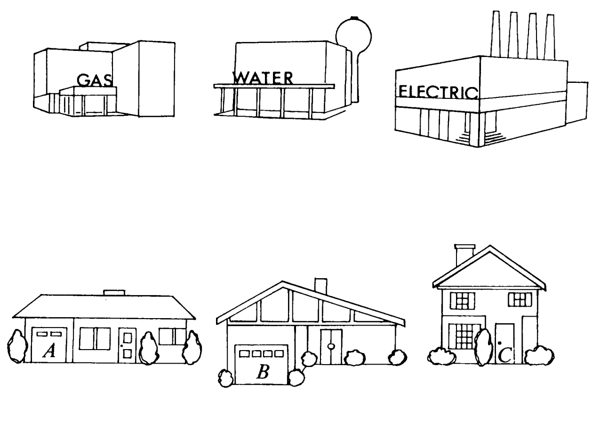



FIGURE 1

**Two equivalent problems. In FIGURE 1 three utilities are to be connected to each of three houses, while in FIGURE 2 the like numbered handles of the urns are to be joined. This latter task would be easy if handles were numbered in the same order on both urns.**

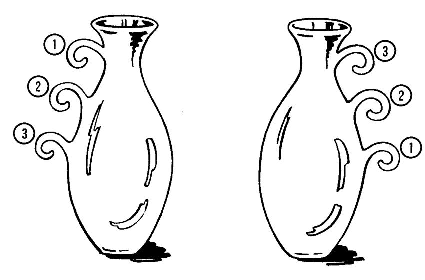



FIGURE 2

and many mathematical historians feel that its origins are at least a century older than that. (Any new information along these lines will be gratefully received.)

Still another problem, whose equivalence with the utility problem is less obvious, concerns a Persian caliph with a beautiful daughter [**6**, pp. 274–276].

> A caliph was so troubled by the large number of suitors, that he decided to set up a competition to determine who was best qualified to marry the girl. In the qualifying round, the aspirants were presented with a picture of two urns, each having three handles numbered 1, 2, and 3 from top to bottom. It was required to join 1 with 1, 2 with 2, and 3 with 3 by curves which do not intersect each other or cross the urns. This task was not so difficult, but a caliph's daughter is not so easily won. The father insisted that any suitor who survived the first round also compete in the finals. This time the same urns were pictured, but the numbers on the handles of the second urn were in reverse order (FIGURE 2).

Relabeling the ends of handles 1 and 3 on the first urn and the end of handle 2 on the second urn as $W$, $G$, and $E$ and the other three handles as $A$, $B$, and $C$, this problem is seen to be equivalent to the utility problem provided we consider the urns themselves as paths joining some of the houses and utilities.

One reason that these problems are perennial favorites among mathematical puzzlists is their deceptive simplicity. At first glance they appear easy to solve, yet it can actually be proved that no solution exists. (It is rumored that the caliph's daughter died as an old maid.) You can begin to appreciate the difficulty as soon as you try to connect the houses and utilities. Although any two of the houses can be connected to all three utilities with no lines crossing, you run into difficulties when you try to connect the third house. Persons who claim to have solved the problem usually resort to a semantic loophole. The owner of one house, for example, can permit a utility company to run its line through his basement on the way to another house. Of course this trick will not work if we think of the houses and utilities as being represented by single points.

Dudeney's solution to the problem in 1917 is illustrated in FIGURE 3. However, he published the problem again in 1926 [**4**, Problem 156; reprinted in **5**, Problem 413]. This time he remarked that he had been receiving an average of ten letters per month from correspondents asking about it. He also gave a proof "for the first time in a book" that there is "no solution without any trick."

Suppose we connect houses $A$ and $B$ with $W$, $G$, and $E$. By changing the location of $B$, we have an equivalent network in the shape of a rhombus with one diagonal as shown in FIGURE 4. This divides the plane into three regions, and, no matter where house $C$ is placed, it will be in some region and one of the utilities $W$, $G$, or $E$ will be outside that region. It will be impossible

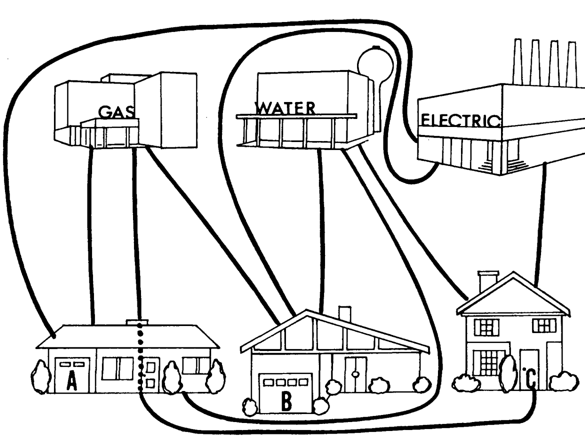



FIGURE 3

**A trick solution to the utilities problem (FIGURE 3) suggests a proof that no legitimate solution is possible. FIGURE 4 represents three utilities (center row) connected to two houses (top and bottom). No matter where the third house is located, it will be impossible to join it with all three utilities without any lines crossing.**

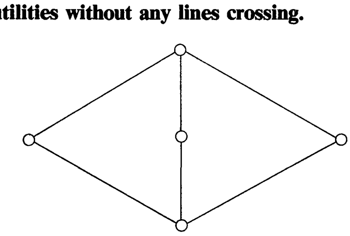
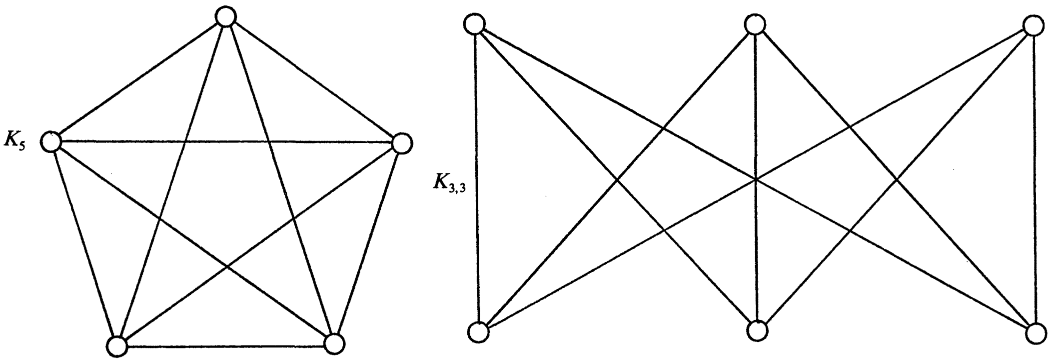


FIGURE 4

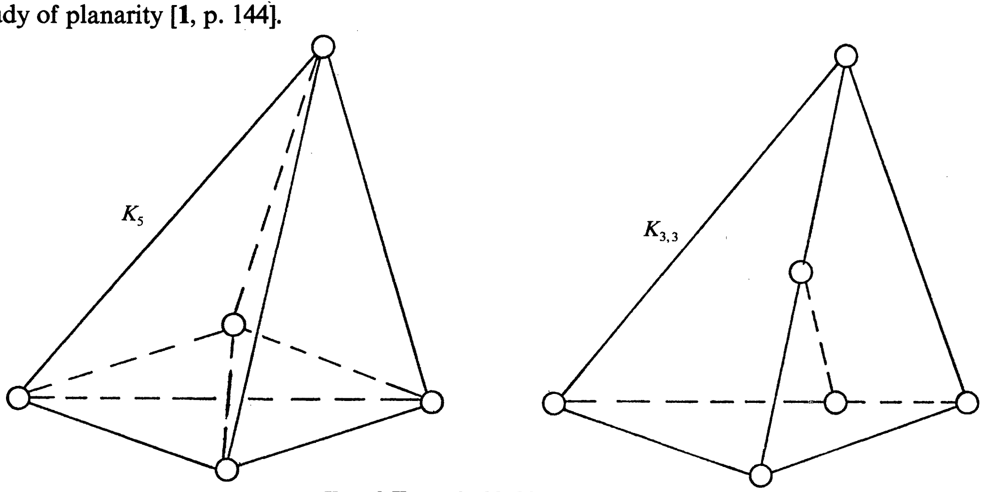
to connect $C$ to that utility without crossing the boundary of the region, and the boundary consists of other utility lines. This proof is based on the Jordan curve theorem, which was proved in 1905.

The utilities problem is especially significant because of its role in the characterization of nonplanar graphs. During the 1920's, a number of mathematicians were searching for criteria that would characterize whether or not a particular graph can be drawn in a plane without intersecting itself. The first such criterion was announced by the Polish mathematician Kazimierz Kuratowski in 1929 and published in 1930. He proved that a graph is planar if it contains no subgraph equivalent to one of the graphs $K_5$ or $K_{3,3}$ (FIGURE 5). Two American mathematicians, Orrin Frink and P. A. Smith, independently arrived at the same result approximately six months after Kuratowski.




**The fundamental non-planar graphs, $K_5$ and $K_{3,3}$.**
FIGURE 5

The graph $K_5$ is a pentagon with all of its diagonals, while $K_{3,3}$ is precisely the graph obtained as a result of making all of the connections required in the utilities problem. Showing that $K_{3,3}$ is nonplanar is equivalent to showing that the utilities problem has no solution. Interestingly enough, Kuratowski did not publish a detailed proof that these two graphs are nonplanar. Instead, he drew the tetrahedral forms of these two graphs (FIGURE 6) and, in that time-honored mathematical tradition, asserted that "obviously neither graph is homeomorphic to a graph located in the plane" [7, p. 272]. Undoubtedly Kuratowski was more concerned with the converse—that these two examples essentially characterize all nonplanar graphs. There is also reason to believe that Kuratowski was not familiar with the utilities problem when he began his study of planarity [1, p. 144].

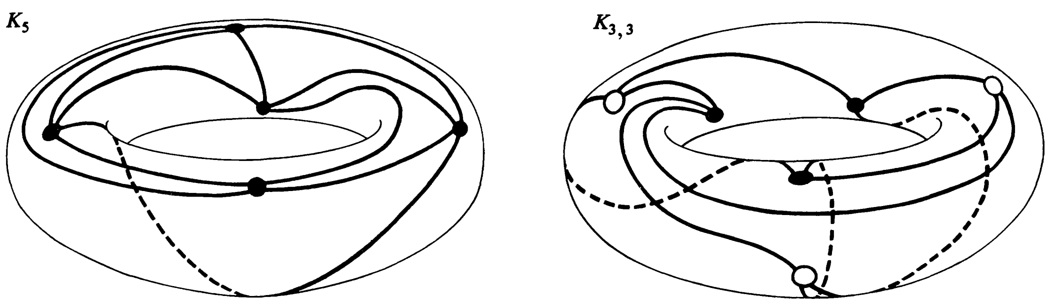



**$K_5$ and $K_{3,3}$ embedded in 3-space.**

FIGURE 6

Another proof that the graphs $K_5$ and $K_{3,3}$ cannot be planar follows from the Euler-Descartes formula $V-E+F=2$, where $V$, $E$, and $F$ represent the numbers of vertices, edges, and faces of a finite, connected, planar graph. (The exterior region is also counted as a face.) In the case of the two graphs in question, it is difficult to count the faces directly, since they overlap. However, it is clear from FIGURE 6 that all the faces of $K_5$ are triangles, while all the faces of $K_{3,3}$ are quadrilaterals. In general, for any planar graph in which every face is an $n$-gon, the total number of edges is given by $E=n(V-2)/(n-2)$. To see this, note that each edge belongs to exactly two faces. So if each face has exactly $n$ edges, then $nF=2E$. The result follows by substitution in the Euler Descartes formula $V-E+F=2$.

Thus for $K_5$, we would have $E=3(5-2)/(3-2)=9$. But, by actual count, $E=10$. Similarly, for $K_{3,3}$ we would have $E=4(6-2)/(4-2)=8$, but, by actual count, $E=9$. Hence neither $K_5$ nor $K_{3,3}$ can be drawn in a plane without any crossing of edges, and they cannot be drawn on a sphere either. However, as FIGURE 7 shows, both graphs can be drawn on a torus. This may be good news for utility companies of the future who wish to lay out their lines on a torus-shaped orbiting space station.




**$K_5$ and $K_{3,3}$ drawn on a torus.**
FIGURE 7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# An Uncharacteristic Proof of the Spectral Theorem


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Linear algebra textbooks usually prove the spectral theorem near the end of the book, following discussion of eigenvalues and eigenvectors. This sequence makes it appear that the proof depends on solving the characteristic equation to find eigenvalues, and thus on the whole circle of ideas concerning determinants and the Cayley-Hamilton Theorem. I would like to offer, as an alternative, an elementary proof that does not depend on eigenvectors, determinants, or the characteristic equation. This proof is essentially a specialization of the proof used for infinite-dimensional Hilbert spaces [**3**, pp. 269–275]. (Richard Brualdi pointed out that similar ideas are used in the proof given by Lange in [**2**, p. 294].) The advantage I claim for this proof is that it uses minimal hypotheses; it clears away the debris of determinants and eigenvalues on which the theorem usually seems to depend. I believe students will do better on the general problem of diagonalization if they first have a clear understanding of the spectral theorem.

We begin with a vector space $V$ of dimension $n$ over the real numbers, with inner product $(u,v)$ and norm $|u|=(u,u)^{1/2}$. Let $T$ be a symmetric linear operator on $V$, i.e., $(Tu,v)=(u,Tv)$ for all $u$ and $v$ in $V$. Our object is to show that $T$ can be decomposed into a sum of the form $r_1E_1+r_2E_2+\cdots+r_kE_k$, where the $E_i$ are orthogonal projections ($E_i^2=E_i$, $E_iE_j=0$ if $i\neq j$) whose sum $E_1+E_2+\cdots+E_k$ is the identity operator $I$. Besides the elementary properties of vector spaces, bases, and inner products, we will use just two special facts to prove this theorem: First the fundamental theorem of algebra, that every polynomial with complex coefficients has complex roots; second, what Lange [**2**, p. 125] calls the fundamental theorem of linear algebra, that a homogeneous system of linear equations has non-trivial solutions if the number of variables exceeds the number of equations.

The first step in the proof is to establish that there is some non-zero polynomial $p(x)=a_0+a_1x+\cdots+a_mx^m$ such that $p(T)=a_0I+a_1T+\cdots+a_mT^m$ is the zero operator. This fact is established quite easily by representing $T$ by a matrix so that the equation $p(T)=0$ becomes a homogeneous system of $n^2$ linear equations in $m+1$ unknowns $a_0,\ldots,a_m$. (Actually because $T$ is symmetric at most $\frac{1}{2}n(n+1)$ of these equations are independent, but this fact is not needed.) If $m=n^2$, there are more variables than equations, and so there is a non-trivial solution. In this way we can find, by a constructive procedure (Gaussian elimination, for instance), a non-zero polynomial $p(x)$ such that $p(T)=0$. (If we wish, we can look successively for solutions with $m=0$, $m=1$, $m=2,\ldots,$ until, for some $m\leqslant n$, we find the minimal polynomial of $T$.)

The next part of the proof is not constructive. Having found the polynomial $p(x)$, we factor it over the real numbers into a product of powers of distinct linear and irreducible quadratic factors. It is an easy consequence of the fundamental theorem of algebra that every polynomial $p(x)$ over the real numbers factors in this way:

$$p(x)=(x-r_1)^{e_1}(x-r_2)^{e_2}\cdots(x-r_k)^{e_k}\left((x-s_1)^2+t_1^2\right)^{f_1}\cdots\left((x-s_j)^2+t_j^2\right)^{f_j}$$

where $r_1,\ldots,r_k$ are the distinct real roots of $p(x)$, $s_1,\ldots,s_j$ are real numbers, $t_1,\ldots,t_j$ are non-zero real numbers, and $e_1,\ldots,e_k,f_1,\ldots,f_j$ are positive integers. Our object is to show that $q(x)=(x-r_1)\cdots(x-r_k)$ satisfies $q(T)=0$. This fact will be established by two separate arguments, one allowing us to eliminate the quadratic factors from $p(x)$, and another allowing us to reduce each exponent $e_i$ to unity.

First let $h(x)=(x-s)^2+t^2$, where $t$ is not zero. We claim $h(T)$ is one-to-one. Suppose $h(T)u=0$. Then

$$\begin{aligned}0=(h(T)u,u)&=((T-sI)^2u,u)+t^2(u,u)\\&=((T-sI)u,(T-sI)u)+t^2|u|^2\\&=|(T-sI)u|^2+t^2|u|^2\\&\geqslant t^2|u|^2.\end{aligned}$$

Since $t$ is not zero, it follows that $u=0$; hence $h(T)$ is one-to-one, as we claimed. It is now easy to see that if $g$ is any polynomial and $(hg)(T)=h(T)g(T)=0$, then $g(T)=0$. Applying this fact to $p(x)=((x-s_1)^2+t_1^2)p_1(x)$, we see that $p_1(T)=0$. By repeating the argument we eventually demonstrate that $T$ is annihilated by the polynomial $r(x)=(x-r_1)^{e_1}\cdots(x-r_k)^{e_k}$.

Our second argument shows that the exponents in $r(x)$ may all be taken to be unity. In fact if $B$ is any symmetric operator, then $B, B^2, B^3, \ldots$, all have the same null space. Indeed if $B^{n+1}u=0$, $n \geq 1$, then $0=(B^{n+1}u, B^{n-1}u)=(B^n u, B^n u)=|B^n u|^2$. Hence $B^n u=0$ also. (Conversely if $B^n u=0$, it is trivial that $B^{n+1}u=0$.) We now apply this result to $B=T-r_1 I$. Since $r(T)=0$, the null space of $B^{e_1}$ contains the vector $(T-r_2 I)^{e_2}\cdots(T-r_k I)^{e_k}u$ for every $u$ in $V$. By what was just shown, all these vectors must be in the null space of $B$. Thus, $T$ is annihilated by the polynomial $(x-r_1)(x-r_2)^{e_2}\cdots(x-r_k)^{e_k}$. By repeating the argument we eventually show that $T$ is annihilated by the polynomial $q(x)=(x-r_1)\cdots(x-r_k)$. (Throughout this proof we will be using the fact that substitution of an operator in a polynomial changes polynomial multiplication to operator composition, i.e., that $(hg)(T)=h(T)g(T)$. Since polynomial multiplication is commutative, we need not be concerned with the order of the factors either before or after substitution.)

At this point standard arguments (such as those used in the primary decomposition theorem [**1**, p. 204]) yield the spectral theorem immediately. If $V_j$ denotes the null space of $T-r_j I$, it is easily seen that $V_j$ and $V_h$ are orthogonal whenever $j$ and $h$ are distinct. In fact if $u$ is in $V_j$ and $v$ is in $V_h$, then

$$r_j(u,v)=(r_j u,v)=(Tu,v)=(u,Tv)=(u,r_h v)=r_h(u,v).$$

Since $r_j$ and $r_h$ are different, $(u,v)=0$. Further, if $q_j(x)=(x-r_1)\cdots(x-r_{j-1})(x-r_{j+1})\cdots(x-r_k)$, then

$$t(x)=\frac{q_1(x)}{q_1(r_1)}+\cdots+\frac{q_k(x)}{q_k(r_k)}-1$$

is a polynomial of degree $k-1$ having $k$ distinct roots, namely $r_1,\ldots,r_k$. Therefore $t(x)=0$ for all $x$, so

$$1=\frac{q_1(x)}{q_1(r_1)}+\cdots+\frac{q_k(x)}{q_k(r_k)}.$$

It follows immediately that

$$u=Iu=\frac{q_1(T)u}{q_1(r_1)}+\cdots+\frac{q_k(T)u}{q_k(r_k)}$$

for all $u$ in $V$. Since $(T-r_j I)q_j(T)=q(T)=0$, each $q_j(T)u$ must belong to $V_j$, the null space of $T-r_j I$. This proves the direct-sum spectral decomposition $V=V_1\oplus\cdots\oplus V_k$, and from it all standard facts of spectral theory are now readily accessible. For instance, if $E_j$ is the projection on $V_j$ along $V_1\oplus\cdots\oplus V_{j-1}\oplus V_{j+1}\oplus\cdots\oplus V_k$, then $E_j E_i=0$ if $j$ and $i$ are distinct. Also $I=E_1+\cdots+E_k$, $T=r_1E_1+\cdots+r_kE_k$, and more generally $t(T)=t(r_1)E_1+\cdots+t(r_k)E_k$ for any polynomial $t(x)$. In particular, if $t_i$ is chosen so that $t_i(r_j)=\delta_{ij}$, then $t_i(T)=E_i$, so that each $E_i$ is a polynomial in $T$.

This proof can easily be adapted to show that a Hermitian operator on a finite-dimensional complex inner product space can be spectrally decomposed. In fact, the proof is even shorter than the one given above, since no irreducible quadratic factors occur when one uses complex numbers. The simultaneous diagonalization of a commuting set of symmetric or Hermitian operators can then be proved in a few minutes. From these two facts and the standard decomposition of a normal operator as a linear combination of commuting Hermitian operators, i.e., $T=(T^*+T)/2+i(iT^*-iT)/2$, one can easily prove the spectral theorem for a normal operator.

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# Placing Counters to Illustrate Burnside's Lemma

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The six different arrangements of 2 counters on a $2\times 2$ board (FIGURE 1) fall naturally into two essentially different groups: $s_1, s_2, s_3$ and $s_4$ are congruent (or essentially alike) via reflections and/or rotations of each other, as are the two diagonal forms $s_5$ and $s_6$. More generally, for positive integers $k, N$ with $k \leqslant N^2$, let $f(k,N)$ denote the number of essentially different arrangements of $k$ counters on an $N\times N$ board. Thus $f(2,2)=2$ and, from FIGURE 2, $f(1,3)=3$, although there are 9 different ways of placing a single counter on a $3\times 3$ board.

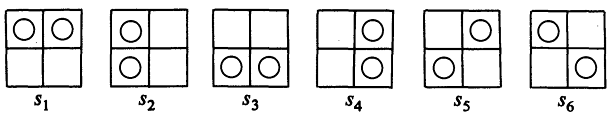


FIGURE 1

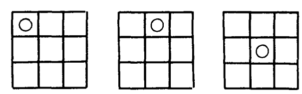

FIGURE 2

In "A pattern recognition problem" [**2**, p. 306] Ray Lipman challenges the reader to devise a computer program that will recognize essentially different arrangements and hence determine $f(k,N)$ for small values of $k$ and $N$. In fact a fairly simple expression for $f(k,N)$ was given [**5**, p. 289] with a suggestion of a derivation using Burnside's Lemma ([**3**, p. 136], [**7**]. In this note we show in some detail how Burnside's Lemma can be used to find $f(k,N)$ and to solve other counting problems.

We first describe the Lemma (in italics), simultaneously carrying along the corresponding simple example of $f(2,2)$.

*Burnside's Lemma. Let $\mathbb{S}=\{s_1,s_2,\ldots,s_m\}$ be a finite set.*

*Example of* $f(2,2)$. $\mathbb{S}=\{s_1,s_2,s_3,s_4,s_5,s_6\}$ is the set of 6 arrangements of 2 counters on a $2\times 2$ board.

*Let $\mathcal{G}=\{E,G_2,G_3,\ldots,G_n\}$ be a finite group each of whose n elements induces a permutation of the elements of $\mathbb{S}$.*

$\mathcal{G}$ is the group of symmetries of the square [**1**, p. 34]. It has 8 elements (see FIGURE 3): the identity $E$ (which leaves every point in the plane fixed); reflections $R_1, R_2$ in the diagonals $r_1, r_2$ of the square; reflections $R_3, R_4$ in the lines $r_3, r_4$ through the center and parallel to a pair of opposite sides; counterclockwise rotations $S_{90}$, $S_{180}$, $S_{270}$ through angles $90°, 180°, 270°$, respectively, about the center $O$ of the square. Each of these 8 elements of $\mathcal{G}$ induces a permutation of the elements of $\mathbb{S}$ e.g., $R_1$ maps $s_1,s_2,s_3,s_4,s_5,s_6$ respectively onto $s_4,s_3,s_2,s_1,s_5,s_6$.

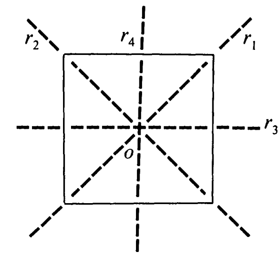


FIGURE 3

*We say $s_i$ is essentially like $s_j$ if there is a (not necessarily unique) element $G \in \mathcal{G}$ such that $s_i = G(s_j)$; of course $s_i$ and $s_j$ are essentially different if $s_i$ is different from $G(s_j)$ for every $G \in \mathcal{G}$.*

The arrangement $s_2$ is essentially like $s_1$ because $s_2 = S_{90}(s_1)$ or $s_2 = R_2(s_1)$; similarly, $s_4$ is essentially like $s_2$ because $s_4 = S_{180}(s_2)$ or $s_4 = R_4(s_2)$. However, $s_5$ and $s_1$ are essentially different because $E(s_1) = R_4(s_1) = s_1, S_{90}(s_1) = R_2(s_1) = s_2, S_{180}(s_1) = R_3(s_1) = s_3$, and $S_{270}(s_1) = R_1(s_1) = s_4$; hence there is no element $G \in \mathcal{G}$ for which $G(s_1) = s_5$.

*Let $\chi(G)$ denote the number of elements of $\mathcal{S}$ fixed by the element $G \in \mathcal{G}$.*

$\chi(E) = 6$ because $E$ fixes all 6 elements of $\mathcal{S}$; $\chi(R_1) = 2$ because $R_1$ fixes $s_5, s_6$. Likewise $\chi(R_2) = \chi(R_3) = \chi(R_4) = 2, \chi(S_{90}) = \chi(S_{270}) = 0$, $\chi(S_{180}) = 2$.

*Then the number of essentially different elements of $\mathcal{S}$ under the action of the elements of $\mathcal{G}$ is $\frac{1}{n}\{\chi(E) + \chi(G_2) + \chi(G_3) + \cdots + \chi(G_n)\}$.*

The number $f(2,2)$ of essentially different arrangements of 2 counters on a $2 \times 2$ board, under action of the symmetries of the square board is

$$\begin{aligned}&\tfrac{1}{8}\{\chi(E) + \chi(R_1) + \chi(R_2) + \chi(R_3) + \chi(R_4)\\ &\qquad + \chi(S_{90}) + \chi(S_{180}) + \chi(S_{270})\}\\ &\qquad = \tfrac{1}{8}\{6+2+2+2+2+0+2+0\} = 2.\end{aligned}$$

"Essentially like" is, of course, an equivalence relation which partitions $\mathcal{S}$ into equivalence classes sometimes called the orbits of $\mathcal{G}$. For more details of equivalence relations see [3] or [7].

We now use Burnside's Lemma to determine $f(k,N)$ in general. Let $\mathcal{S}$ consist of the $\binom{N^2}{k}$ arrangements of $k$ counters on an $N \times N$ board, and let $\mathcal{G}$ be the group of 8 symmetries of the square board. Of course $\chi(E) = \binom{N^2}{k}$.

To determine $\chi(R_1)$ we must place $k$ counters so that the reflection $R_1$ maps the arrangement onto itself. Suppose we place $i$ of the counters on squares cut (diagonally) by $r_1$, which can be done in $\binom{N}{i}$ ways. The remaining $k-i$ counters must be placed in pairs which are congruent by the reflection $R_1$, so $k-i$ must be even and we can place $(k-i)/2$ counters on any of $(N^2-N)/2$ squares below the diagonal $r_1$; the positions of the other $(k-i)/2$ counters are then determined. Thus

$$\chi(R_1) = \sum_{i \equiv k \pmod 2} \binom{N}{i}\binom{(N^2-N)/2}{(k-i)/2}.$$

Clearly $\chi(R_2) = \chi(R_1)$.

To determine $\chi(R_3)$ consider first the situation when $N$ is even. The counters must be placed in pairs congruent by reflection in $r_3$, so $k$ must be even and we place $k/2$ of them on $k/2$ of the $N^2/2$ squares below $r_3$. The positions of the remaining $k/2$ are then determined, so

$$\chi(R_3)=\begin{cases}\binom{N^2/2}{k/2} & \text{if } N \text{ is even and } k \text{ is even,}\\ 0 & \text{if } N \text{ is even and } k \text{ is odd.}\end{cases}$$

When $N$ is odd a count similar to the one for $\chi(R_1)$ shows that

$$\chi(R_3)=\sum_{i\equiv k \pmod 2}\binom{N}{i}\binom{(N^2-N)/2}{(k-i)/2} \quad \text{if } N \text{ is odd.}$$

Clearly, $\chi(R_4)=\chi(R_3)$.

Let us now turn to the counting of $\chi(S_{90})$. If $N$ is even the lines $r_3$ and $r_4$ partition the $N\times N$ board into 4 congruent $N/2\times N/2$ quarter-squares and the placing of counters in just one of these quarters determines the placing in the others (if the arrangement is to be counted in $\chi(S_{90})$). Hence

$$\chi(S_{90})=\begin{cases}\binom{N^2/4}{k/4} & \text{if } N \text{ is even and } k\equiv 0 \pmod 4,\\ 0 & \text{if } N \text{ is even and } k\not\equiv 0 \pmod 4.\end{cases}$$

On the other hand, if $N$ is odd the board has a central square and $N^2-1$ non-central squares, and the counters on the non-central squares come in sets of 4 that are cyclically permuted by $S_{90}$. Hence

$$\chi(S_{90})=\begin{cases}\binom{(N^2-1)/4}{k/4} & \text{if } N \text{ odd, } k\equiv 0 \pmod 4,\\ \binom{(N^2-1)/4}{(k-1)/4} & \text{if } N \text{ odd, } k\equiv 1 \pmod 4,\\ 0 & \text{if } N \text{ odd, } k\equiv 2,3 \pmod 4.\end{cases}$$

Finally, a similar argument leads to

$$\chi(S_{180})=\begin{cases}\binom{N^2/2}{k/2} & \text{if } N \text{ even, } k \text{ even,}\\ 0 & \text{if } N \text{ even, } k \text{ odd.}\\ \binom{(N^2-1)/2}{k/2} & \text{if } N \text{ odd, } k \text{ even,}\\ \binom{(N^2-1)/2}{(k-1)/2} & \text{if } N \text{ odd, } k \text{ odd.}\end{cases}$$

Putting this all together by Burnside's Lemma we deduce the following values of $8f(k,N)$: If $N$ is even,

$$8f(k,N)=\begin{cases}A+2B+3C+2D, & \text{if } k\equiv 0 \pmod 4,\\ A+2B+3C, & \text{if } k\equiv 2 \pmod 4,\\ A+2B; & \text{if } k\equiv 1 \text{ or } 3 \pmod 4;\end{cases}$$

if $N$ is odd,

$$8f(k,N)=\begin{cases}A+4B+C+2D, & \text{if } k\equiv 0 \text{ or } 1 \pmod 4,\\ A+4B+C, & \text{if } k\equiv 2 \text{ or } 3 \pmod 4.\end{cases}$$

where

$$A=\binom{N^2}{k},\quad B=\sum_{i\equiv k(\text{mod }2)}\binom{N}{i}\binom{(N^2-N)/2}{(k-i)/2},\quad C=\binom{[N^2/2]}{[k/2]},\quad D=\binom{[N^2/4]}{[k/4]},$$

and [ ] denotes the greatest integer function.

We present several further applications of Burnside's Lemma.

*Example* 1. Given a regular polygon with $p$ sides, where $p$ is a prime, and $m$ differently colored paints, we can color the $p$ sides in $m^p$ ways. In other words, there are $m^p$ $m$-colored regular $p$-gons. Call two such colored $p$-gons essentially alike if they are related by a rotation. How many essentially different $m$-colored $p$-gons are there?

Here $\mathcal{S}$ is the set of $m^p$ $m$-colored $p$-gons. $\mathcal{G}$ is the cyclic group with $p$ elements: the identity and $p$-1 rotations through angles $2\pi/p, 2(2\pi/p), 3(2\pi/p), \ldots, (p-1)(2\pi/p)$. The identity fixes all $m^p$ elements of $\mathcal{S}$ while each rotation fixes just the $m$ monochromatic $p$-gons. Hence the number of essentially different elements of $\mathcal{S}$ is

$$\frac{1}{p}\{m^p+m+m+\cdots+m\}=\frac{1}{p}\{m^p+(p-1)m\}.$$

Since this number is an integer, $m^p+(p-1)m\equiv 0 \pmod p$, or $m^p\equiv m \pmod p$. If $m$ is not divisible by $p$, this yields $m^{p-1}\equiv 1 \pmod p$, the well-known "Little Fermat theorem" [**6**, p. 95]. A generalization, which can also be deduced using Burnside's Lemma, is that if $m$ and $n$ are relatively prime positive integers, then $m^{\phi(n)}\equiv 1 \pmod n$, where $\phi(n)$ is the number of positive integers less than $n$ that are relatively prime to $n$ [**10**, p. 147].

*Example* 2. Let $Q_1, Q_2, \ldots, Q_p$ be the $p$ vertices of a regular $p$-gon, where $p$ is prime. Corresponding to a permutation $(\alpha_1, \alpha_2, \ldots, \alpha_{p-1})$ of $\{1,2,\ldots,p-1\}$ there is a "directed" $p$-gon obtained by drawing the directed line segments $Q_pQ_{\alpha_1}, Q_{\alpha_1}Q_{\alpha_2}, \ldots, Q_{\alpha_{p-1}}Q_p$. For example, FIGURE 4 exhibits three directed $p$-gons. In general we can draw $(p$-1)! directed $p$-gons. Under the action of the group of rotations of the regular $p$-gon, how many essentially different directed $p$-gons are there?

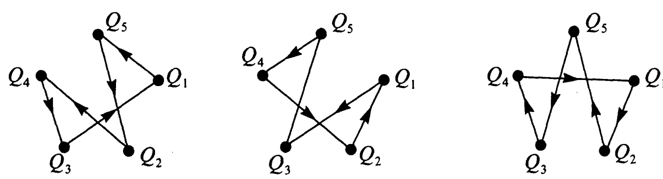


FIGURE 4

By Burnside's Lemma there are $(1/p)((p-1)!+(p-1)(p-1))$ essentially different directed $p$-gons. Hence $(p-1)!\equiv -1 (\text{mod } p)$. This is the well-known Wilson's Theorem of elementary number theory [**10**, p. 153].

*Example* 3. It is interesting to note that we can combine the two previous examples into one [**4**]. Observe first that of the $(p-1)!$ directed $p$-gons, $p-1$ are regular since a regular $p$-gon will be determined once the first edge $Q_pQ_{\alpha_1}$ is taken. Now suppose we $m$-color (the $p$ edges of) all the directed $p$-gons. Clearly there are $m^p(p-1)!$ $m$-colored directed $p$-gons. A rotation fixes the $m(p-1)$ regular monochromatic $p$-gons. Hence the number of essentially different $m$-colored directed $p$-gons is $(1/p)(m^p(p-1)!+(p-1)m(p-1))$. Consequently,

$$m^p(p-1)!\equiv m(p-1)(\text{mod } p).$$

Setting $m=1$ gives Fermat's Theorem $(p-1)!\equiv p-1 \pmod p$, while dividing on the left by $(p-1)!$ and on the right by $p-1$ yields Wilson's Theorem.

*Example* 4. Suppose $1 \leqslant M < N$. Then $k$ counters can be placed on an $M \times N$ board in $\binom{MN}{k}$ ways, many of which are essentially alike because of reflection and/or rotation of the rectangular board. How many essentially different arrangements are there?

Let $\mathbb{S}$ be the set of $\binom{MN}{k}$ arrangements, and let $\mathcal{G}$ be the group of symmetries of the rectangle. $\mathcal{G}$ has 4 elements: the identity; two reflections, $R_1$ and $R_2$ in lines $r_1$ and $r_2$ respectively; and the rotation $S$ of 180° about the center of the rectangle (FIGURE 5).

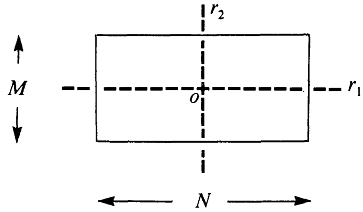


FIGURE 5

If we let $f(k,M,N)$ denote the number of essentially different arrangements of the $k$ counters on the $M \times N$ board, then

$$4f(k,M,N)=\begin{cases} A+3B & \text{if } M \text{ is even, } N \text{ is even, } k \text{ is even,}\\ A & \text{if } M \text{ is even, } N \text{ is even, } k \text{ is odd,}\\ A+2B+C & \text{if } M \text{ is even, } N \text{ is odd, } k \text{ is even,}\\ A+C & \text{if } M \text{ is even, } N \text{ is odd, } k \text{ is odd,}\\ A+2B+D & \text{if } M \text{ is odd, } N \text{ is even, } k \text{ is even,}\\ A+D & \text{if } M \text{ is odd, } N \text{ is even, } k \text{ is odd,}\\ A+C+D+E & \text{if } M \text{ is odd, } N \text{ is odd, } k \text{ is even,}\\ A+C+D+F & \text{if } M \text{ is odd, } N \text{ is odd, } k \text{ is odd,} \end{cases}$$

where

$$A=\binom{MN}{k},\quad B=\binom{MN/2}{k/2},\quad C=\sum_{i\equiv k \pmod 2}\binom{M}{i}\binom{(N-1)M/2}{(k-i)/2},$$

$$D=\sum_{i\equiv k \pmod 2}\binom{N}{i}\binom{(M-1)N/2}{(k-i)/2},\quad E=\binom{(MN-1)/2}{k/2},\quad F=\binom{(MN-1)/2}{(k-1)/2}.$$

**[8]** and **[9]** contain related material on Burnside's Lemma and Pólya's enumeration method.


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# Powers of Sums of Digits

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Let $G(A)$ be the sum of the squares of the digits in the integer $A$. Then, denoting by $G^n(A)$ the result of $n$ successive applications of the operator $G$ to $A$, A. Porges proved (in [3]) that for any number $A$ there exists either a positive integer $n$ such that $G^n(A)=1$ or a positive integer $m$ such that $G^m(A)=4$. Since the set of eight numbers 4, 16, 37, 58, 89, 145, 42 and 20 is closed under the operation $G$, every natural number is eventually transformed by $G$ either to unity or to one of these eight numbers. Kiyosi Iseki generalised Porges' problem in [2] showing that sums of cubes produce the following cyclic sequences:

(i) Self closed sequences with length one are 1, 153, 370, 371 and 407.
(ii) Cyclic sequences with length two are 136, 244; and 919, 1459.
(iii) Cyclic sequences with length three are 55, 250, 133; and 160, 217, 352.

B. M. Stewart [4] also generalised the paper by Porges in several directions.

In this note we reverse the calculation, studying not sums of powers of digits but powers of sums of digits. Specifically, let $N$ be a natural number and let $g_r(N)$ denote the $r$th power of the sum of the digits of $N$. Let $g_r^i(N)=g_r(g_r^{i-1}(N))$, where $g_r(N)=g_r^1(N)$. We shall determine for $r=1,2,\dots,10$ the integers $N$ that satisfy $g_r^i(N)=N$ for minimal values of $i$.

Suppose the number of digits in $N$ is $n$. Then for some positive integer $\delta$, we have $10^{\delta-1}<n\leqslant 10^{\delta}$, Since each digit is less than or equal to 9, we have $g_r(N)\leqslant(9\cdot 10^{\delta})^r$, implying that $g_r(N)$ has at most $r(\delta+1)$ digits. In general, if the number of digits in $g_r^{i-1}(N)$ is $\leqslant 10^{\theta}$, then the number of digits in $g_r^i(N)$ is at most $r(\theta+1)$. For each $r\leqslant 10$ it can be shown that there exists a term $g_r^{i-1}(N)$ having not more than $10^2$ digits, implying a $g_r^i(N)$ having at most $r(2+1)=3r$ digits. We shall consider the cases $r=2,3,4$ in detail and give a table of results for all $r\leqslant 10$.

Consider $r=2$. It is not difficult to show that after a finite number of steps a term $g_2^i(N)$ will be obtained which has at most 4 digits. Suppose a term $g_2^i(N)$ has at least 5 digits, i.e., $10^{k-1}\leqslant g_2^i(N)<10^k$, where $k-1\geqslant 4$. Then $g_2^i(N)$ has $k$ digits and $g_2^{i+1}(N)\leqslant(9k)^2=81k^2$. By induction we see easily that $81k^2<10^{k-1}$ for $k-1\geqslant 4$, implying that the sequence decreases: $g_2^{i+1}(N)<g_2^i(N)$. Therefore a term $M$ in the sequence must eventually be reached which has at most 4 digits.

For such an $M$ we have $g_2^{i+1}(N)=g_2(M)\leqslant(9\cdot 4)^2=36^2=1296$. Now the terms $g_2^i(N)$ are all squares, and for the squares up to 1296, none has a sum of digits exceeding $9+9+9=27$. Thus

$$g_2^{i+2}(N)=g_2^2(M)\leqslant \quad 27^2 \quad =729,$$

and similarly

$$g_2^3(M)\leqslant(6+9+9)^2=576,$$

$$g_2^4(M)\leqslant(4+9+9)^2=484,$$

$$g_2^5(M)\leqslant(3+9+9)^2=441.$$

By direct trial of the 21 cases of squares up to 441, we find that each sequence contains a term equal to either 1, 81, 169 or 256. Once one obtains a 1 or an 81, these terms repeat indefinitely since $g_2(1)=1$ and $g_2(81)=81$. Also, because $g_2(169)=256$ and $g_2(256)=169$, each of the terms 169 and 256 yield an alternating sequence of 169's and 256's. We have, then, the solutions $N=169=13^2$ and $N=256=16^2$ of $g_2^2(N)=N$ (see TABLE 1).

| $r=2$ | | $r=3$ | | $r=4$ | |
|---|---|---|---|---|---|
| $i=1$ | $i=2$ | $i=1$ | $i=2$ | $i=1$ | $i=2$ |
| $1^2$ | $13^2$ | $1^3$ | $19^3$ | $1^4$ | $18^4$ |
| $9^2$ | $16^2$ | $8^3$ | $28^3$ | $7^4$ | $27^4$ |
| | | $17^3$ | | $22^4$ | |
| | | $18^3$ | | $25^4$ | |
| | | $26^3$ | | $28^4$ | |
| | | $27^3$ | | $36^4$ | |

**Solutions to $g_r^i(N)=N$: all integers which generate cyclic patterns under repeated application of the operation $g_r(N)$ which raises the sum of the digits of $N$ to the $r$th power.**

TABLE 1

Consider $r=3$. Suppose a term $g_3^i(N)$ has at least 7 digits, i.e., $10^{k-1} \leqslant g_3^i(N) < 10^k$, where $k-1 \geqslant 6$. Then $g_3^i(N)$ has $k$ digits and $g_3^{i+1}(N) \leqslant (9k)^3 = 729k^3$. But by induction we see easily that $729k^3 < 10^{k-1}$ for $k-1 \geqslant 6$, implying that the sequence decreases: $g_3^{i+1}(N) < g_3^i(N)$. Therefore a term $M$ in the sequence must eventually appear which has at least 6 digits. For such an $M$ we have $g_3^{i+1}(N) = g_3(M) \leqslant (9.6)^3 = 157464$.

Now $g_3^i(N)$ are all cubes and for the cubes up to 157464, none has a sum of digits exceeding $9+9+9+9+9=45$. Thus

$$g_3^{i+2}(N) = g_3^2(M) \leqslant \quad 45^3 \quad = 91125,$$

and similarly

$$g_3^3(M) \leqslant (8+9+9+9+9)^3 = 85184,$$

$$g_3^4(M) \leqslant (7+9+9+9+9)^3 = 79507,$$

$$g_3^5(M) \leqslant (7+8+9+9+9)^3 = 74008,$$

$$g_3^6(M) \leqslant (6+9+9+9+9)^3 = 74008.$$

By direct trial of 42 cases of cubes up to 74088, we find that each sequence contains a term equal to one of the integers 1, 512, 4913, 5832, 6859, 17576, 19683 or 21952. Because $g_3(6859)=21952$ and $g_3(21952)=6859$ each of the terms 6859 and 21952 yield an alternating sequence of 6859's and 21952's; the other terms repeat indefinitely. (These other terms are the cubes of 1, 8, 17, 18, 26 and 27, reported in 1879 by Moret Blanc to be the only numbers equal to the sum of the digits of their cubes.) TABLE 1 contains a summary of all solutions to $g_3^i(N)=N$.

| $r=5$ | | | $r=6$ | $r=7$ | | | $r=8$ | | $r=9$ | | $r=10$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $i=1$ | $i=2$ | $i=4$ | $i=1$ | $i=1$ | $i=2$ | $i=4$ | $i=1$ | $i=4$ | $i=1$ | $i=4$ | $i=1$ | $i=3$ | $i=6$ |
| $N=1^5$ | $23^5$ | $7^5$ | $1^6$ | $1^7$ | $38^7$ | $36^7$ | $1^8$ | $31^8$ | $1^9$ | $45^9$ | $1^{10}$ | $43^{10}$ | $45^{10}$ |
| $28^5$ | $29^5$ | $22^5$ | $18^6$ | $18^7$ | $44^7$ | $54^7$ | $46^8$ | $52^8$ | $54^9$ | $72^9$ | $82^{10}$ | $61^{10}$ | $63^{10}$ |
| $35^5$ | $31^5$ | $25^5$ | $45^6$ | $27^7$ | $46^7$ | $63^7$ | $54^8$ | $67^8$ | $71^9$ | $90^9$ | $85^{10}$ | $70^{10}$ | $72^{10}$ |
| $36^5$ | $34^5$ | $40^5$ | $54^6$ | $31^7$ | $47^7$ | $72^7$ | $63^8$ | $70^8$ | $81^9$ | $99^9$ | $94^{10}$ | | $81^{10}$ |
| $46^5$ | | | $64^6$ | $34^7$ | $55^7$ | | | | | | $97^{10}$ | | $90^{10}$ |
| | | | | $43^7$ | $56^7$ | | | | | | $106^{10}$ | | $99^{10}$ |
| | | | | $53^7$ | $62^7$ | | | | | | $117^{10}$ | | |
| | | | | $58^7$ | $65^7$ | | | | | | | | |
| | | | | $68^7$ | | | | | | | | | |

**Solutions of $g_r^i(N)=N$ for $5 \leqslant r \leqslant 10$.**

TABLE 2

Now we consider $r=4$. Arguing as before we can show that after a finite number of steps, a term $g_4^i(N)$ will be obtained which has at most 8 digits. If $M$ has 8 digits then $g_4(M) \leqslant 72^4$. Then $g_4^2(M) \leqslant 64^4$ whence $g_4^3(M) \leqslant 63^4$. By direct trial of the 63 cases of fourth powers up to $63^4$ we find that each sequence contains a term equal to either 1, 2401, 104976, 234256, 390625, 531441, 614656 or 1679616. Excepting for $N=531441$ and 104976, we have $g_4(N)=N$ for all these terms. Since $g_4(531441)=104976$ and $g_4(104976)=531441$ we have the solutions $N=104976=18^4$ and $N=531441=27^4$ of $g_4^2(N)=N$ (see TABLE 1).

Other solutions to the equation $g_r^i(N)=N$ for powers $r$ greater than 4 are listed in TABLE 2.

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# Renumbering of the Faces of Dice


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The February 1978 issue of *Scientific American* reports (on page 19) that George Sicherman discovered a way to number two cubic dice in a different manner from the usual, yet with the same sum probabilities as two standard cubic dice. In this note, we shall relate this result to cyclotomic polynomials, and use this relationship to extend Sicherman's scheme to the faces of $k$ dice formed from the Platonic solids. For further related results, see [**2**].

Abstractly, a die with faces numbered $n_1,\dots,n_f$ is equivalent to an urn containing $f$ balls labelled $n_1,\dots,n_f$, respectively. (The advantage of treating the problem in terms of urns is that the geometric properties of the die, which are irrelevant to the solution, need not be considered.) The key idea in our analysis will be a one-to-one correspondence between urns and polynomials of a certain type. Suppose $U$ is an urn containing a finite number of balls, each labelled with a positive integer. To $U$ we associate the polynomial $U(x)=\sum_{i=1}^{\infty} b_i x^i$, where $b_i$ is the number of balls labelled $i$. Conversely, if $V(x)=\sum_{i=1}^{\infty} a_i x^i$ is a polynomial with non-negative integer coefficients, only a finite number of which are non-zero, we can get an urn $V$ whose associated polynomial is $V(x)$ by placing $a_i$ balls labelled $i$ into $V$. (Since $V(x)/V(1)$ is exactly the generating function for the probability distribution determined by $V$, the results in this note can be viewed in part, as a special case of the fact (see [**1**], p. 236) that the probability generating function for a sum of independent distributions is the same as the product of the probability generating functions for the individual distributions.)

We shall now characterize those sets of urns containing $n$ balls which have the same sum probabilities as a set of standard $n$-urns, where by a **standard $n$-urn** we mean an urn containing $n$ balls labelled 1 through $n$, inclusive, whose associated polynomial is

$$\sum_{i=1}^{n} x^i = \left(\frac{x^n-1}{x-1}\right)x.$$

Let $A_1,\dots,A_k$ be urns containing $n$ balls apiece, each ball labelled by a positive integer. Let $B_1,\dots,B_r$ be $r$ standard $n$-urns. Let $P_A[S=t]$ be the probability that when one ball is chosen from each of $A_1,\ldots,A_k$, the sum of the labels on the chosen balls is $t$; $P_B[S=t]$ is defined similarly for the urns $B_1,\ldots,B_r$.

The first important property of these sum probabilities is that if $P_A[S=r]=P_B[S=r]$, then $r=k$. If each urn contains exactly one ball, the conclusion is obvious. So let's assume that $n>1$. Since the balls in the urns are labelled with positive integers, $P_A[S=t]\neq 0$ only when $t\geqslant k$. On the other hand, if $P_A[S=t]\neq 0$ then $P_A[S=t]\geqslant 1/n^k$, since there are $n^k$ ways to choose one ball from each of $A_1,\ldots,A_k$. As $P_A[S=r]=P_B[S=r]=1/n^r\neq 0$, we have $k\leqslant r$ from the first condition and $1/n^r\geqslant 1/n^k$ from the latter. Therefore $r=k$, as required.

Now suppose each set of urns yields identical sum probabilities; in other words, assume that $P_A[S=t]=P_B[S=t]$ for all $t$. Let $A_i(x)$ be the polynomial associated with urn $A_i$: the coefficient of $x^j$ in $A_i(x)$ is the number of balls labelled $j$ in urn $A_i$. Then, as we shall show,

(a) $A_i(0)=0, A_i(1)=n$,
(b) all coefficients of $A_i(x)$ are non-negative integers,
(c) the coefficient of $x$ in $A_i(x)$ is one, and,
(d) $\prod_{i=1}^{k} A_i(x)=\left(\frac{(x^n-1)x}{x-1}\right)^k.$

Moreover, the converse is also true: if $A_1(x),\ldots,A_k(x)$ are any polynomials which satisfy (a)–(d), then the set consisting of the urns $A_1,\ldots,A_k$ whose associated polynomials are $A_1(x),\ldots,A_k(x)$,

$$x(x^4+x^3+x^2+x+1)(x+1)^a(x^2+1)^b(x^4-x^3+x^2-x+1)^c(x^8-x^6+x^4-x^2+1)^d$$

| | a b c d | Numbering |
|---|---|---|
| $I_1$ | 1 1 1 1 | 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 |
| $I_2$ | 1 1 0 0 | 8 7 7 6 6 6 5 5 5 5 4 4 4 4 3 3 3 2 2 1 |
| $I_3$ | 1 1 2 2 | 32 28 27 24 23 22 20 19 18 17 16 15 14 13 11 10 9 6 5 1 |
| $I_4$ | 1 1 0 1 | 16 15 15 14 14 13 13 12 12 11 6 5 5 4 4 3 3 2 2 1 |
| $I_5$ | 1 1 2 1 | 24 22 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 3 1 |
| $I_6$ | 1 1 1 0 | 12 11 10 10 9 9 8 8 7 7 6 6 5 5 4 4 3 3 2 1 |
| $I_7$ | 1 1 1 2 | 28 27 24 23 20 19 18 17 16 15 14 13 12 11 10 9 6 5 2 1 |
| $I_8$ | 0 2 0 1 | 17 16 15 15 14 14 13 13 12 11 7 6 5 5 4 4 3 3 2 1 |
| $I_9$ | 2 0 2 1 | 23 22 19 18 18 17 15 14 14 13 11 10 10 9 7 6 6 5 2 1 |
| $I_{10}$ | 0 2 0 2 | 25 24 23 22 21 15 15 14 14 13 13 12 12 11 11 5 4 3 2 1 |
| $I_{11}$ | 2 0 2 0 | 15 14 13 12 11 10 10 9 9 8 8 7 7 6 6 5 4 3 2 1 |
| $I_{12}$ | 0 2 1 1 | 21 19 19 17 17 15 15 13 13 11 11 9 9 7 7 5 5 3 3 1 |
| $I_{13}$ | 2 0 1 1 | 19 18 18 17 15 14 14 13 11 10 10 9 7 6 6 5 3 2 2 1 |
| $I_{14}$ | 0 2 1 2 | 29 27 25 23 21 19 19 17 17 15 15 13 13 11 11 9 7 5 3 1 |
| $I_{15}$ | 2 0 1 0 | 11 10 10 9 9 8 8 7 7 6 6 5 5 4 4 3 3 2 2 1 |
| $I_{16}$ | 1 1 2 0 | 16 14 14 12 12 11 10 10 9 9 8 8 7 7 6 5 5 3 3 1 |
| $I_{17}$ | 0 2 1 3 | 37 33 29 27 27 25 23 23 21 19 19 17 15 15 13 11 11 9 5 1 |
| $I_{18}$ | 2 0 0 0 | 7 6 6 6 5 5 5 5 4 4 4 4 3 3 3 3 2 2 2 1 |
| $I_{19}$ | 2 0 3 1 | 27 23 22 22 19 18 18 17 15 14 14 13 11 10 10 9 6 6 5 1 |
| $I_{20}$ | 0 2 0 0 | 9 8 7 7 7 6 6 6 5 5 5 5 4 4 4 3 3 3 2 1 |
| $I_{21}$ | 0 2 1 0 | 13 11 11 11 9 9 9 9 7 7 7 7 5 5 5 5 3 3 3 1 |
| $I_{22}$ | 2 0 3 0 | 19 17 15 14 14 13 12 12 11 10 10 9 8 8 7 6 6 5 3 1 |

**Icosahedral Dice**

TABLE 5

respectively, has the same sum probabilities as a set of $k$ standard $n$-urns.

It is clear from the definition of $A_i(x)$ that if $P_A[S=t]=P_B[S=t]$ for all $t$, then (a) and (b) will be satisfied. The number of ways to choose one ball from each of the $A_i$ and get a sum of $t$ is the same as the coefficient of $x^t$ in $\prod_{i=1}^{k} A_i(x)$. Thus, $\prod_{i=1}^{k} A_i(x)=n^k\sum_t P_A[S=t]x^t$. Similarly, $[(x^n-1)x/(x-1)]^r=n^r\sum_t P_B[S=t]x^t$. Since, as we showed above, $r=k$, (d) is established. By comparing terms of lowest degree in (d) and recalling that $A_1(x),\dots,A_k(x)$ have no constant terms, (c) follows. The converse follows immediately, since the polynomials determine the sum probabilities: $P_A[S=t]=P_B[S=t]$ for all $t$, where $B_1,\dots,B_k$ are $k$ standard $n$-urns.

Now suppose that $D$ is an $n$-faced die which can be combined with other $n$-faced dice to form a set of $k$ dice which has the same sum probabilities as a set of standard $n$-faced dice. The theorem shows that the associated polynomial $D(x)$ is a factor of $[(x^n-1)x/(x-1)]^k$. It is well known that $(x^n-1)/(x-1)$ is the product of the $d$th cyclotomic polynomials where $d$ ranges over the divisors of $n$ with $d>1$ and that the cyclotomic polynomials are irreducible over the rationals. (See, for example [3], pp. 362–363.) Thus $D(x)$ is of the form

$$x\prod_{d|n,\,d>1} F_d(x)^{a_d}, \tag{1}$$

where $F_d(x)$ is the $d$th cyclotomic polynomial, $a_d$ is a non-negative integer, and the product is taken over all integers $d>1$ which divide $n$.

$x(x+1)^a(x^2+1)^b$

| | $a\ b$ | Numbering |
|---|---|---|
| $T_1$ | 1 1 | 4 3 2 1 |
| $T_2$ | 2 0 | 3 2 2 1 |
| $T_3$ | 0 2 | 5 3 3 1 |

**Tetrahedral Dice**

$x(x+1)(x^2+x+1)(x^2-x+1)^a$

| | $a$ | Numbering |
|---|---|---|
| $C_1$ | 1 | 6 5 4 3 2 1 |
| $C_2$ | 0 | 4 3 3 2 2 1 |
| $C_3$ | 2 | 8 6 5 4 3 1 |

**Cubic Dice**

$x(x+1)^a(x^2+1)^b(x^4+1)^c$

| | $a\ b\ c$ | Numbering |
|---|---|---|
| $O_1$ | 1 1 1 | 8 7 6 5 4 3 2 1 |
| $O_2$ | 2 1 0 | 5 4 4 3 3 2 2 1 |
| $O_3$ | 0 1 2 | 11 9 7 7 5 5 3 1 |
| $O_4$ | 2 0 1 | 7 6 6 5 3 2 2 1 |
| $O_5$ | 0 2 1 | 9 7 7 5 5 3 3 1 |
| $O_6$ | 1 2 0 | 6 5 4 4 3 3 2 1 |
| $O_7$ | 1 0 2 | 10 9 6 6 5 5 2 1 |
| $O_8$ | 3 0 0 | 4 3 3 3 2 2 2 1 |
| $O_9$ | 0 3 0 | 7 5 5 5 3 3 3 1 |
| $O_{10}$ | 0 0 3 | 13 9 9 9 5 5 5 1 |

**Octahedral Dice**

TABLES 1–3

$x(x^2+x+1)(x+1)^a(x^2+1)^b(x^2-x+1)^c(x^4-x^2+1)^d$

| | $a\ b\ c\ d$ | Numbering |
|---|---|---|
| $D_1$ | 1 1 1 1 | 12 11 10 9 8 7 6 5 4 3 2 1 |
| $D_2$ | 1 1 0 0 | 6 5 5 3 3 3 3 3 3 2 2 1 |
| $D_3$ | 1 1 2 2 | 18 15 14 12 11 10 9 8 7 5 4 1 |
| $D_4$ | 1 1 0 1 | 10 9 9 8 8 7 4 3 3 2 2 1 |
| $D_5$ | 1 1 2 1 | 14 12 11 10 9 8 7 6 5 4 3 1 |
| $D_6$ | 1 1 1 0 | 8 7 6 6 5 5 4 4 3 3 2 1 |
| $D_7$ | 1 1 1 2 | 16 15 12 11 10 9 8 7 6 5 2 1 |
| $D_8$ | 0 2 0 1 | 11 10 9 9 8 7 5 4 3 3 2 1 |
| $D_9$ | 2 0 2 1 | 13 12 10 9 9 8 6 5 5 4 2 1 |
| $D_{10}$ | 0 2 0 2 | 15 14 13 9 9 8 8 7 7 3 2 1 |
| $D_{11}$ | 2 0 2 0 | 9 8 7 6 6 5 5 4 4 3 2 1 |
| $D_{12}$ | 0 2 1 1 | 13 11 11 9 9 7 7 5 5 3 3 1 |
| $D_{13}$ | 2 0 1 1 | 11 10 10 9 7 6 6 5 3 2 2 1 |
| $D_{14}$ | 0 2 1 2 | 17 15 13 11 11 9 9 7 7 5 3 1 |
| $D_{15}$ | 2 0 1 0 | 7 6 6 5 5 4 4 3 3 2 2 1 |
| $D_{16}$ | 1 1 2 0 | 10 8 8 7 6 6 5 5 4 3 3 1 |
| $D_{17}$ | 0 2 1 3 | 21 17 15 15 13 11 11 9 7 7 5 1 |
| $D_{18}$ | 2 0 0 0 | 5 4 4 4 3 3 3 3 2 2 2 1 |
| $D_{19}$ | 2 0 3 1 | 15 12 12 11 9 8 8 7 5 4 4 1 |
| $D_{20}$ | 0 2 0 0 | 7 6 5 5 5 4 4 3 3 3 2 1 |
| $D_{21}$ | 0 2 1 0 | 9 7 7 7 5 5 5 5 3 3 3 1 |
| $D_{22}$ | 2 0 3 0 | 11 9 8 8 7 6 6 5 4 4 3 1 |

**Dodecahedral Dice**

TABLE 4

TABLES 1 – 5 give, for $n=4, 6, 8, 12, 20$, respectively, all polynomials of form (1) which satisfy conditions (a)—(c). The numberings of the faces of the die which are associated with each polynomial are also given. In the tables the standard die is listed first, followed by pairs of dice which have the same sum probabilities as two standard dice. In TABLES 3–5 some unpaired dice remain. To see that every polynomial corresponds to a die which is contained in some set with the same sum probabilities as a set of standard dice, observe that $\{O_8, O_9, O_{10}\}$ has the same sum probabilities as three copies of $O_1$. Each of the sets $\{D_{16}, D_{17}, D_{18}\}$, $\{D_2, D_{10}, D_{19}\}$, $\{D_3, D_{13}, D_{20}\}$, $\{D_7, D_{13}, D_{21}\}$, and $\{D_4, D_{10}, D_{22}\}$ has the same sum probabilities as three copies of $D_1$. Finally, three copies of $I_1$ have the same sum probabilities as each of $\{I_{16}, I_{17}, I_{18}\}$, $\{I_2, I_{10}, I_{19}\}$, $\{I_3, I_{13}, I_{20}\}$, $\{I_7, I_{13}, I_{21}\}$, and $\{I_4, I_{10}, I_{22}\}$.

There do remain several open questions. It is a straightforward, though tedious, exercise to determine which polynomials satisfy the conditions of the theorem, provided $n$ is small. However, for arbitrary $n$, there remains the problem of determining, other than by direct computation, all polynomials with non-negative coefficients which are products of cyclotomic polynomials. With the aid of the tables, one could find all ways to combine non-standard Platonic dice to get sets with the same sum probabilities as sets of standard Platonic dice. (We leave this as an exercise for the reader.) Note that if not all dice in the set have the same number of faces, some very strange sets emerge. For example, $\{T_1, O_1, D_1, I_1\}$ has the same sum probabilities as $\{T_3, O_4, D_{13}, I_{12}\}$ and two copies of $T_1$ have the same probabilities as $\{P_1, O_6\}$, where $P_1$ is a fair coin whose sides are numbered 1 and 2.

The author wishes to thank the referees and the editors for their suggestions.

**References**


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[3] I. N. Herstein, Topics in Algebra, 2nd ed., Wiley, New York, 1975.


# Elementary (My Dear Watson) Differential Equation


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It was hot for this London fall day—70°F. Holmes arrived at Barker Street Annex to find the inspector hunched over the body. "It is important that we determine the exact time of death, sir, for in that way we may immediately determine the motive," said the inspector. Not wishing to pursue the unpursuable non sequitur, Holmes took out his r----- thermometer and after a few moments of discrete (!) investigation, announced, "I say! 94.6°F. (What, no metric system?) And it is presently noon." With that he departed into the London fog, to return to the body at the same spot in one hour. After performing another investigation Holmes declared, "93.4°F at 1 o'clock." And then silence....

"Inspector, the murder occurred at exactly 8:58.51204 o'clock a.m. Good day to you, sir!"

Later in their chambers Watson asked, "I say! (The British always say by saying, unlike the Hsitirb's who say by not saying.) How did you do that, Holmes?" "Elementary, my dear Watson. A simple application of a law of Newton," said Holmes.

"Here, here Holmes, you don't mean to say he fell out of a window?" snapped Watson, sensing the gravity of the issue. Remaining *cool* Holmes began at the beginning, where he always began with Watson, "You see... ."

# PROBLEMS

DAN EUSTICE, Editor
LEROY F. MEYERS, Associate Editor
*The Ohio State University*

## Proposals

*To be considered for publication, solutions should be mailed before June 1, 1980.*

**1080.** Some calculators have an "int" key. The "integral part of $x$" is given by $\text{int}\,x=[|x|]\text{sgn}\,x$, where $[x]$ denotes the greatest integer not greater than $x$ and where $\text{sgn}\,x$ is $-1$ when $x<0$, 0 when $x=0$, and $+1$ when $x>0$.

We have $|x|=x\,\text{sgn}\,x$ and $\max(x,y)=(x+y+|x-y|)/2$ as examples of familiar functions which can be expressed in terms of "sgn" together with the operations $\{+,-,\times,\div\}$. Show that these functions can be similarly expressed in terms of "int." [*Marlow Sholander, Case Western Reserve University.*]

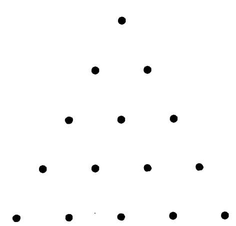
**1081.** Find all real $t$ such that for all $x>y>0$,

$$(x-y)^t(x+y)^t=(x^t-y^t)^t(x^t+y^t)^{2-t}.$$

[*Edwin P. McCravy, Midlands Technical College.*]

**1082.** Prove that $\tan\sin x>\sin\tan x$ for $0<x<\pi/2$. [*C. S. Gardner, University of Texas.*]

**1083.** Given an equilateral point lattice with $n$ points on a side, it is easy to draw a polygonal path of $n$ segments passing through all the $n(n+1)/2$ lattice points. Show that it cannot be done with less than $n$ segments. [*M. S. Klamkin and A. Liu, University of Alberta.*]



$n=5$

ASSISTANT EDITORS: DON BONAR, *Denison University;* WILLIAM A. MCWORTER, JR., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgment of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

**1084.** In the game of Kriegspiel Hex two players sit back to back, each with his own Hex board. (For a description and discussion of the game of Hex see Chapter 8 of *The First Book of Mathematical Puzzles and Diversions* by Martin Gardner, Simon and Schuster (1959).) An umpire with a master board directs the game as each player attempts to make a legal move without seeing his opponent's move. The umpire's duties are: (1) Advise each player of his turn, following a legal move by his opponent. (2) Declare an illegal move so that the offending player can try a different move. (3) State when a player has won.

(a) Show that there is a winning strategy for the first player in Kriegspiel Hex played on a $3\times 3$ board.

(b) Prove that there is no winning strategy for the first player in Kriegspiel Hex played on an $n\times n$ board, $n\geqslant 4$. [*William A. McWorter, Jr., The Ohio State University.*]

**1085.** Consider the polynomial $P(x)=x^4-14x^2+x+38$. Find a function $g=g(x;\epsilon_1,\epsilon_2)$, where $\epsilon_1$ and $\epsilon_2$ are $\pm 1$, such that the recursive sequence $x_{n+1}=g(x_n)$ converges to a different zero of $P(x)$ for each of the four distinct values of $(\epsilon_1,\epsilon_2)$. [*Bert Waits, The Ohio State University.*]

**1086.** Consider the following transformations on $4\times 4$ matrices. Let $R$ move the top row to the bottom and the other rows cyclically up; let $D$ be the reflection across the main diagonal; let $S$ be the interchange of the 1st and 2nd rows followed by the interchange of the 1st and 2nd columns. What is the order of the group generated by $R,D$, and $S$? [*Barbara Turner, California State University, Long Beach.*]

**1087.** Let $\sum_{k=-\infty}^{k=+\infty} a_k z^k$ be the Laurent series of $e^{z+\frac{1}{z}}$ for $0<|z|<\infty$.

(a) Show that each $a_k$ is an irrational number.

(b) Show that the set $\{a_k : k\geqslant 0\}$ is linearly dependent over the rationals. [*Barbara Turner, California State University, Long Beach.*]

**1088.** (a) For each positive integer $m$, how many Pythagorean triangles are there which have an area equal to $m$ times the perimeter? How many of these are primitive?

(b*) Can this result be generalized to all triangles with integer sides and area equal to $m$ times the perimeter? [*Alan Wayne, Pasco-Hernando Community College.*]

# Quickies

*Solutions to Quickies appear at the conclusion of the Problems section.*

**Q663.** Prove that $|1+z|\leqslant|1+z|^2+|z|$ for complex $z$. [*Anon, Erewhon-upon-Spanish River.*]

**Q664.** Prove that

$$\sum_{k=1}^{n}(x_k+1/x_k)^a\geqslant\frac{(n^2+1)^a}{n^{a-1}} \qquad (1)$$

where $x_k>0$ $(k=1,2,\ldots,n)$, $a>0$ and $x_1+x_2+\cdots+x_k=1$. [*M. S. Klamkin, University of Alberta.*]

# Solutions

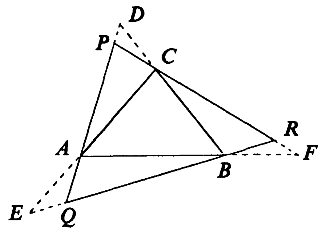
**Two Triangles** **May 1977**

**1014.** Given a triangle, $\triangle ABC$, points $D, E$, and $F$ are on the lines determined by $BC, CA$, and $AB$, respectively. The lines $AD, BE$, and $CF$ intersect to form triangle $\triangle PQR$.

(a) Show that $\triangle PQR$ is equilateral if and only if $\triangle ABC$ is.

(b) Express the area of $\triangle PQR$ in terms of the area of $\triangle ABC$. [*K. R. S. Sastry, Addis Ababa, Ethiopia.*]

The missing hypothesis that $AD = BE = CF$ was added on page 221 of the September 1977 issue.




*Solution*: (a) *Case* I. Suppose $D, E$, and $F$ are on the line segments $BC, CA$, and $AB$, respectively, and $\triangle ABC$ is equilateral. $AD^2 = AB^2 + BD^2 - AB \cdot BD$ and $BE^2 = BC^2 + CE^2 - BC \cdot CE$. The hypotheses and these two equations yield that either $BD = CE$ or $BD + CE = BC$. But the latter is impossible unless $P = Q = R$. Now $BD = CE$ implies angle $BAD =$ angle $CBE$ which shows that angle $QPR = 60°$. Similarly, angle $PRQ =$ angle $RQP = 60°$.

Now suppose $\triangle PQR$ is equilateral and $AD = BE = CF$. Then, with $a(\triangle T)$ denoting the area of triangle $T$, we have

$$\frac{RP}{AD} = \frac{a(\triangle BRP)}{a(\triangle ABD)} = \frac{a(\triangle CRP)}{a(\triangle ACD)} = \frac{a(\triangle BRP) + a(\triangle PQR) + a(\triangle CPQ)}{a(\triangle ABC)}$$

and similar results for $PQ/BE$ and $QR/CF$. Since these ratios are equal, it follows that $a(\triangle BRP) = a(\triangle CPQ) = a(\triangle AQR)$ and then that $BP = CQ = AR$ and thus $\triangle ABC$ is equilateral.

*Case* II. When $D, E$, and $F$ are on $BC, CA$, and $AB$ produced, the proof follows a similar line with slight variations.

(b) As a special case of Routh's theorem [this MAGAZINE, January 1976, pp. 25–27], we have in Case I that

$$a(\triangle PQR) = \frac{|BD - DC|^2 a(\triangle ABC)}{BD^2 + BD \cdot DC + DC^2} = \frac{4DD^{*2} a(\triangle ABC)}{AD^2}$$
$$= \frac{4AD^2 - 2AB^2}{AD^2} a(\triangle ABC) = \left(4 - \frac{3AB^2}{AD^2}\right) a(\triangle ABC),$$

where $D^*$ is the midpoint of $BC$. In Case II,

$$a(\triangle PQR) = \frac{(BD + DC)^2 a(\triangle ABC)}{BD^2 - BD \cdot DC + DC^2}$$

also reduces to $(4 - 3AB^2/AD^2) a(\triangle ABC)$.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*Also solved by J. M. Stark.*

**Fibonacci Sets** **March 1978**

**1037.** Let $n$ be an integer greater than 2 and let $A_1, A_2, \ldots, A_n$ be non-empty sets of positive integers with the property that $a \in A_i$ and $b \in A_{i+1}$ implies $a+b \in A_{i+2}$, where we identify $A_{n+1}$ as $A_1$ and $A_{n+2}$ as $A_2$.

(a) If $1 \in A_1$ and $2 \in A_2$, find an integer that belongs to at least two of the sets.

(b)* Is is possible for $A_1, A_2, \ldots, A_n$ to be pairwise disjoint? [*James Propp, Great Neck, New York.*]

*Solution of* (a): Let $F_n$ be the $n$th Fibonacci number. Then $F_2 = 1 \in A_1, F_3 = 2 \in A_2$, and $F_{i+1} \in A_i$. Thus $F_2 + F_{n+1} \in A_2$, $F_2 + (F_2 + F_{n+1}) \in A_3, F_3 + (F_2 + F_{n+1} + F_2) \in A_4$, and continuing, we find $m = F_{n-1} + (F_2 + F_{n+1} + F_3 + F_4 + \cdots + F_{n-2}) \in A_n$. Furthermore, if $n > 4$, we find, since $F_{n+2} \in A_1$, that $F_3 + F_{n+2} \in A_3$, $F_3 + (F_3 + F_{n+2}) \in A_4$, and finally that $k = F_{n-3} + (F_{n-4} + \cdots + F_3 + F_{n+2}) \in A_{n-2}$. Since $m - k$ is $F_2 + F_2 + F_{n+1} - F_{n+2} - F_3 + F_{n+2} + F_{n-1} = 0$, we have a common element in $A_n$ and $A_{n-2}$ if $n > 4$.

It is quickly verified that for $n = 3$ that 5 belongs to $A_1$ and $A_3$ and for $n = 4$ we see that 10 belongs to both $A_2$ and $A_3$.

GRAHAM LORD
Université Laval, Quebec

*Solution of* (b): The sets $A_1 = \{x : x = 1 + 5k\}, A_2 = \{x : x = 3 + 5k\}, A_3 = \{x : x = 4 + 5k\}$, and $A_4 = \{x : x = 7 + 5k\}$ with $k = 0, 1, 2, 3, \ldots$ are easily seen to satisfy the hypotheses and are pairwise disjoint. Collections of pairwise disjoint sets for $n = 4, 5, 7, 8, 9, \ldots, 15$ have been constructed but a general existence theorem has not been proved. For $n = 3$, it can be shown that at least two of the sets $A_1, A_2$, and $A_3$ are not disjoint.

JAMES D. WATSON
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*Also solved by Eli Leon Isaacson. Part (a) was also solved by Rolf Sonntag (Germany) and the proposer. Part (b) was also solved by Stanley Rabinowitz.*

The problem as published had a misprint with $A_1$ in place of $A_i$. Counterexamples to the misstated problem were supplied by Robert Baer, Nick Franceschine III, Furman University Problem Group, William Myers, and Howard W. Pullman.

**Traces** **March 1978**

**1038.** Define the following sequence of square matrices:

$$M(1) = [1], \qquad M(2) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \qquad M(3) = \begin{bmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix}, \ldots .$$

Find the sum of the elements on the main diagonal of $M(n)$. [*Douglas Lewan, Antrim, New Hampshire.*]

*Solution*: The first entry of $M(n)$ is one more than the sum of the first $n-1$ squares: $1 + n(n-1)(2n-1)/6$. The diagonal elements of $M(n)$ form an arithmetic progression with common difference $n+1$. Thus the sum is $n(n+1)(2n^2 - 2n + 3)/6$.

LENNY KUPER
Notre Dame High School
Cape Girardeau, Missouri

*Also solved by 118 others.*

**1043.** If $(a_i, b_i, c_i)$ are the sides, $R_i$ the circumradii, $r_i$ the inradii, and $s_i$ the semi-perimeters of two triangles $(i=1,2)$, show that

$$\left\{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\right\}^{1/2} \geqslant 3\left\{\frac{1}{\sqrt{a_1a_2}}+\frac{1}{\sqrt{b_1b_2}}+\frac{1}{\sqrt{c_1c_2}}\right\} \tag{1}$$

with equality iff the two triangles are equilateral.

Also show that the analogous three triangle inequality

$$\left\{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\frac{s_3}{r_3R_3}\right\}^{1/2} \geqslant 9\left\{\frac{1}{\sqrt{a_1a_2a_3}}+\frac{1}{\sqrt{b_1b_2b_3}}+\frac{1}{\sqrt{c_1c_2c_3}}\right\} \tag{2}$$

is invalid. [*M. S. Klamkin, University of Alberta.*]

*Solution*: We prove first that

$$\text{(A)} \qquad (a_1+b_1+c_1)(a_2+b_2+c_2) \geqslant 3\left(\sqrt{a_1a_2b_1b_2}+\sqrt{b_1b_2c_1c_2}+\sqrt{a_1a_2c_1c_2}\right).$$

Apply the Schwarz inequality $x_1y_1+x_2y_2 \leqslant \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$ to $L \equiv \sqrt{a_1+b_1+c_1}\sqrt{a_2+b_2+c_2}$ to obtain

$$L \geqslant \sqrt{a_1a_2}+\sqrt{b_1b_2}+\sqrt{c_1c_2} \quad \text{and}$$

$$L^2 \geqslant a_1a_2+b_1b_2+c_1c_2+2\left(\sqrt{a_1a_2b_1b_2}+\sqrt{a_1a_2c_1c_2}+\sqrt{b_1b_2c_1c_2}\right).$$

But certainly for positive $x$, $y$, $z$ we know that $x^2+y^2+z^2 \geqslant xy+xz+yz$. Set $x=\sqrt{a_1a_2}$, $y=\sqrt{b_1b_2}$, and $z=\sqrt{c_1c_2}$ in this inequality, and apply it in the previous expression for $L^2$. This gives (A) immediately.

Now $L^2$ is $4s_1s_2$, and we have the well-known relation $4r_iR_i=(a_ib_ic_i)/s_i$, $(i=1,2)$. Hence

$$4s_1s_2=\sqrt{s_1s_2/(r_1R_1r_2R_2)}\cdot\sqrt{a_1a_2b_1b_2c_1c_2}\,.$$

By using this in (A) and dividing both sides by $\sqrt{a_1a_2b_1b_2c_1c_2}$, (1) is proved. Equality will hold in (A) iff we have $a_1=a_2=b_1=b_2=c_1=c_2$; that is, if both triangles are equilateral, since only then will equality hold in both of the inequalities that were used above.

By using $4r_iR_i=a_ib_ic_i/s_i$, equation (2) becomes analogous to (A):

$$p_1p_2p_3 \geqslant 9\left(\sqrt{a_1a_2a_3b_1b_2b_3}+\sqrt{a_1a_2a_3c_1c_2c_3}+\sqrt{b_1b_2b_3c_1c_2c_3}\right).$$

This is not always true for triangles. Consider sides $a_1$, $a_1/100$, $99a_1/100$ as $a_1$, $b_1$, $c_1$; $a_2$, $a_2/100$, $99a_2/100$ as $a_2$, $b_2$, $c_2$; $a_3$, $a_3/100$, $99a_3/100$ as $a_3$, $b_3$, $c_3$. The left hand side is $8a_1a_2a_3$. The right side is bigger than $(8.8)a_1a_2a_3$.

PAUL BRACKEN, undergraduate
University of Toronto

*Comment*: As a companion inequality, we also have

$$4\left\{\frac{1}{\sqrt{a_1}}+\frac{1}{\sqrt{b_1}}+\frac{1}{\sqrt{c_1}}\right\}\left\{\frac{1}{\sqrt{a_2}}+\frac{1}{\sqrt{b_2}}+\frac{1}{\sqrt{c_2}}\right\} \geqslant \left\{\frac{s_1}{r_1R_1}\cdot\frac{s_2}{r_2R_2}\right\}^{1/2}$$

or, equivalently,

$$\text{(B)} \qquad \left\{2\Sigma\sqrt{a_1b_1}\right\}\left\{2\Sigma\sqrt{a_2b_2}\right\} \geqslant \{\Sigma a_1\}\{\Sigma a_2\}.$$

The latter follows from the area of a triangle of sides $a^{1/4}$, $b^{1/4}$, $c^{1/4}$ being non-negative, i.e.,

$$2\Sigma\sqrt{ab}-\Sigma a \geqslant 0.$$

There's equality in (B) if both triangles are degenerate (each having a vanishing side).

M. S. Klamkin
University of Alberta

*Also solved by William McGovern, J. M. Stark, Alexandras Zujus, and the proposer.*

**Irrational** **September 1978**

**1048.** Let $\{a_k\}$ be an increasing sequence of positive integers with $a_{k+1}/a_k \to 1$ as $k\to\infty$. Prove that $\Sigma_{k=1}^{\infty}(a_k-1)^2/a_1\cdots a_k$ is irrational. What happens if it is not assumed that $a_{k+1}/a_k\to 1$ but the series converges? I have some partial results but no complete discussion. [*Paul Erdös, Hungarian Academy of Science.*]

*Solution*: We have that

$$\sum_{k=1}^{N}\frac{(a_k-1)^2}{a_1a_2\cdots a_k}=\sum_{k=1}^{N}\frac{a_k-1}{a_1a_2\cdots a_{k-1}}-\sum_{k=1}^{N}\frac{a_{k+1}-1}{a_1a_2\cdots a_k}+\sum_{k=1}^{N}\frac{a_{k+1}-a_k}{a_1a_2\cdots a_k}$$

$$=a_1-1-\frac{a_{N+1}-1}{a_1a_2\cdots a_N}+\sum_{k=1}^{N}\frac{a_{k+1}-a_k}{a_1a_2\cdots a_k}. \qquad (1)$$

Since $a_{n+1}/a_n\to 1$ as $n\to\infty$, given $\epsilon>0$ there exists an $N(\epsilon)$ such that $(a_{k+1}-a_k)/a_k<\epsilon$ for all $k>N(\epsilon)$. Now suppose

$$\sum_{k=1}^{\infty}\frac{a_{k+1}-a_k}{a_1a_2\cdots a_k}=\frac{m}{n} \text{ for integers } m \text{ and } n. \qquad (2)$$

Then

$$ma_Na_{N-1}\cdots a_1=n\sum_{k=1}^{N}(a_{k+1}-a_k)a_Na_{N-1}\cdots a_{k+1}+n\sum_{k=N+1}^{\infty}\frac{a_{k+1}-a_k}{a_ka_{k-1}\cdots a_{N+1}}.$$

Thus this last summation must be an integer. But

$$n\sum_{k=N+1}^{\infty}\frac{a_{k+1}-a_k}{a_ka_{k-1}\cdots a_{N+1}}<n\epsilon\sum_{k=N+1}^{\infty}\frac{1}{a_ka_{k-1}\cdots a_{N+1}}<n\sum_{1}^{\infty}\frac{1}{(a_{N+1})^k}<1$$

if $N>N(\epsilon)$ and $\epsilon<1/n$. This contradiction shows that the sum in (2) must be irrational. Since the term $(a_{N+1}-1)/a_1a_2\cdots a_N$ in (1) has limit 0, we see that this establishes the result.

Robert Clark, student
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*Also solved by the proposer.*

**Catalan Numbers** **September 1978**

**1049.** For nonnegative integers $n$, let $L_n=\binom{2n}{n}/(n+1)$. Prove that $\Sigma_{k=0}^{n}L_kL_{n-k}=L_{n+1}$. [*Edward T. H. Wang, Wilfred Laurier University.*]

*Solution*: If we set $F(x)=\Sigma_{n=0}^{\infty}L_nx^n$, then $F^2(x)=\Sigma_{n=0}^{\infty}x^n\Sigma_{k=0}^{n}L_kL_{n-k}$, so we need only show that $F^2(x)=\Sigma_{n=0}^{\infty}L_{n+1}x^n$ to reach the desired conclusion. Now if we let $f(x)=$

$d(xF(x))/dx$, we have $f(x)=\sum_{n=0}^{\infty}\binom{2n}{n}x^n$, and it is easily shown (below) that $f(x)=(1-4x)^{-\frac{1}{2}}$. Therefore

$$xF(x)=\int_0^x f(t)\,dt=\tfrac{1}{2}(1-\sqrt{1-4x}\,).$$

Hence

$$F^2(x)=\left[1-2\sqrt{1-4x}+(1-4x)\right]/4x^2 \quad \text{or} \quad F^2(x)=[F(x)-1]/x.$$

Since $L_0=1$, $F(x)-1=\sum_{n=1}^{\infty}L_n x^n$, so that

$$F^2(x)=(1/x)\sum_{n=1}^{\infty}L_n x^n=\sum_{n=0}^{\infty}L_{n+1}x^n.$$

To show $f(x)=(1-4x)^{-\frac{1}{2}}$, we first note that

$$(n+1)\binom{2n+2}{n+1}=(4n+2)\binom{2n}{n},$$

so that

$$\sum_{n=0}^{\infty}(n+1)\binom{2n+2}{n+1}x^n=4x\sum_{n=1}^{\infty}n\binom{2n}{n}x^{n-1}+2\sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

Hence $f'(x)=4xf'(x)+2f(x)$, or $f'(x)/f(x)=2/(1-4x)$. The solution to this differential equation (with initial condition $f(0)=1$) is given by $f(x)=(1-4x)^{-\frac{1}{2}}$.

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Lewis and Clark College

*Also solved by John L. Baker (Canada), Philip M. Benjamin (Taiwan), Theodore S. Bolis, Paul Bracken (Canada), C. S. Davis (England), Nick Francescine III, James O. Friel, Ralph Garfield, Ann Goodsell, Eli L. Isaacson, Hans Kappus (Switzerland), Graham Lord (Canada), Dana Lee Mabbott, P. E. Nuesch (Switzerland), Reinhard Razen (Austria), J. M. Stark, Philip Todd, Michael Vowe (Switzerland), Nikolay Williams, Paul Zwier, and the proposer. There was one unsigned solution. Several solvers provided references to Catalan Numbers in several books on Combinatorics. Goodsell referred to Chapter 3 of W. Feller, An Introduction to Probability Theory and its Application, vol. 1, Wiley, New York, 1950.*

## Differential Equation — September 1978

**1050.** Consider the differential equation $y''+P_1(x)y'+P_2(x)y=0$, where $P_1$ and $P_2$ are polynomials not both constant. Show that this equation has at most one solution of the form $x^a e^{mx}$ for real $a$. [*W. R. Utz, University of Missouri, Columbia.*]

*Solution*: If $y=x^a e^{mx}$, then $y'=(ax^{a-1}+ax^a)e^{mx}$, and $y''=(a(a-1)x^{a-2}+2amx^{a-1}+m^2x^a)e^{mx}$. Substituting in the differential equation and multiplying by $x^{2-a}$ we have

$$x^2(m^2+P_1(x)m+P_2(x))+x(2am+aP_1(x))+a(a-1)=0. \tag{1}$$

Every coefficient of powers of $x$ in (1) is zero. Therefore $a(a-1)=0$ implies $a=0$ or $a=1$. Thus solutions to the differential equation are of the form $e^{mx}$ or $xe^{mx}$. By comparing the coefficients of powers of $x$ in (1), we see that if $P_1(x)$ were constant, then $P_2(x)$ would be constant; consequently $P_1(x)$ is not constant.

*Case* I. Two solutions of the form $e^{mx}$ and $e^{nx}$. Since then $a=0$, $m^2+mP_1(x)+P_2(x)=0$ and thus $(m^2-n^2)+(m-n)P_1(x)=0$. Hence $m=n$.

*Case* II. Two solutions of the form $xe^{mx}$ and $xe^{nx}$. Since then $a=1$, $x(m^2+mP_1(x)+P_2(x))+2m+P_1(x)=0$ and thus $x(m^2-n^2+(m-n)P_1(x))+2(m-n)=0$. Hence $m=n$.

*Case* III. Two solutions of the form $xe^{mx}$ and $e^{nx}$. Then we find that $x(m^2-n^2+(m-n)P_1(x))+2m+P_1(x)=0$. Solving for $P_1(x)$, we must have $m=n$ and thus $P_1(x)$ is constant, which is impossible.

JOHN AMPE, student
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Oak Park, Illinois

*Also solved by Ed Adams, James B. Anderson, Theodore S. Bolis, Paul Bracken (Canada), Thomas E. Elsner, Michael Finn, Eli L. Isaacson, Hans Kappus (Switzerland), Peter W. Lindstrom, R. M. Mathsen (Canada), Zane C. Motteler, Harry Sedinger, J. M. Stark, and the proposer. There were two unsigned solutions.*

# Answers

*Solutions to the Quickies which appear near the beginning of the Problems section.*

**Q663.** We have $|z|=|(1+z)-1|\geqslant -|1+z|+1$ so that $|1+z|^2-|1+z|+|z|\geqslant|1+z|^2-2|1+z|+1=(|1+z|-1)^2\geqslant 0$.

**Q664.** The inequality is an immediate consequence of Jensen's inequality for convex functions $F$, i.e.,

$$\sum_{k=1}^{n} F(x_k)/n \geqslant F\left\{\sum_{k=1}^{n} x_k/n\right\}$$

with equality iff $x_k=$ constant and followed by Cauchy's inequality:

$$\sum_{k=1}^{n}(x_k+1/x_k)^a \geqslant n\left\{\sum_{k=1}^{n}(x_k+1/x_k)/n\right\}^a \geqslant \frac{(n^2+1)^a}{n^{a-1}}.$$

It now remains to show that the function $y=(x+1/x)^a$ is convex in the given domain $0<x<1$ or equivalently that $y''\geqslant 0$. Here,

$$y''=a(x+1/x)^{a-2}\left\{a(1-1/x^2)^2+1/x^4+4/x^2-1\right\}.$$

Since $1/x>1$, $y''\geqslant 0$ for $a\geqslant 0$. It also follows that $y''<0$ for $0>a\geqslant -1$. Consequently, inequality (1) is reversed for this latter domain of $a$. More generally,

$$\sum_{k=1}^{n} F(x_k+1/x_k)/n \geqslant F\left\{\sum_{k=1}^{n}(x_k+1/x_k)/n\right\} \geqslant F(n+1/n)$$

for convex increasing $F$ and the same domain as (1) for the outer inequality.

*Comment*: Inequality (1) is given and proved in D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, pp. 282–283, in a longer way using Lagrange multipliers.

# REVIEWS

**Paul J. Campbell, Editor**
*Beloit College*

**Pierre Malraison, Editor**
*Control Data Corp.*

*Assistant Editor: Eric S. Rosenthal, Princeton University. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Some reviews of books are adapted from the Telegraphic Reviews in the American Mathematical Monthly.*

Calder, Allan, *Constructive mathematics,* Scientific American 241:4 (October 1979) 146-171, 186.

As a philosophy, constructive mathematics maintains that to prove that a mathematical object exists, it must be shown how the object can be constructed. As a program, constructive mathematics attempts to offer "constructive" proofs of as much of "standard" mathematics as it can. This article carefully explains the history of constructivism in mathematics and its latest developments at the hand of Errett Bishop.

May, Robert M., et al., *Management of multispecies fisheries,* Science 205 (20 July 1979) 267-277.

Recent proposals to harvest krill, which are also the food of the baleen whales, provides the occasion for mathematical modeling to analyze harvesting regimes where both predator and prey are taken, and even more complicated systems (e.g., krill-whales-seals, and krill-cephalopods-sperm whales).

Pontryagin, L.S., *Soviet anti-Semitism: Reply by Pontryagin,* Science 205 (14 Sept. 1979) 1083-1084.

Pontryagin rejects charges of anti-Semitism published in an earlier article in *Science* (15 Dec. 1978, pp. 1167-1170). A brief reply to Pontryagin by Alex Rosenberg appeared subsequently (26 Oct., p. 404).

Kolata, Gina Bari, *Mathematicians amazed by Russian's discovery,* Science 206 (2 Nov. 1979) 545-546.

Account of the discovery by L.G. Khachian of a new algorithm for linear programming based on a sequence of ellipsoids whose centers converge on a solution. Khachian's algorithm, which was explicated by Laslo Lovász and Peter Gács, is the first algorithm for LP problems that is provably polynomial: the well known simplex algorithm may grow exponentially as the number of variables increases. (A popularization of Kolata's report appeared subsequently in the *New York Times* and other newspapers on November 11.)

Steen, Lynn Arthur, *Linear programming: A solid new algorithm,* Science News 116 (6 October 1979) 234-236.

An earlier and more general account of the Khachian algorithm than that noted above from *Science*.

Gardner, Martin, *Mathematical games*, Scientific American 241 (October 1979) 18-26.

A discussion of best packing schemes for squares in squares.

Bachmann, Tibor and Bachmann, Peter J., *An analysis of Bêla Bartôk's music through Fibonaccian numbers and the goldean mean*, Musical Quarterly 65 (1979) 72-82.

Offers examples in Bartôk's music where the golden section and Fibonacci numbers are used as a structural foundation, and notes that he was "greatly fascinated by pine cones and sunflowers."

Chandrasekhar, S., *Beauty and the quest for beauty in science*, Physics Today 32:7 (July 1979) 25-30.

In science beauty often serves as a clue to truth. Chandrasekhar, who was responsible for the facsimile edition of Ramanujan's notebooks, explores examples of beauty in science, from Ramanujan's identities to Einstein's theory of relativity and its predictions about black holes. What is it in these discoveries that excites our aesthetic sense? He arrives at two elements: strangeness in proportion, and conformity among the parts.

Spiegel, Richard, *The geometry of psychology*, The Humanist 39 (Jan.-Feb. 1979) 36-38.

The author cites non-euclidean geometries as a parable for psychologists, whom he urges to get out of the rut of "common sense" thought, and to no longer limit their thinking to "stories that have pictures." The editors of *The Humanist* invite reader response.

Penrose, Roger, *Einstein's vision and the mathematics of the natural world*, The Sciences 19 (March 1979) 6-9.

"The problem that confronted the student Einstein, of selecting what is physically important in mathematics from the rest, is now present to a greater degree than ever it was."

Bennett, Judson, *The Poisson distribution: Science pursues a pattern that shapes all we do*, Science Digest 85 (February 1979) 76-78.

Readers of the popular scientific press are accustomed to some confusion of concepts (binomial and Poisson distributions), inaccuracy (a thoroughly butchered equation), and hype ("an astonishing science which...might eradicate chance from our daily lives"). High quality and a popular approach *are* compatible, however, as the article following Bennett's shows (G.B. Kolata of *Science* on exercise and heart disease). Publication of Bennett's article shows that there *is* a forum for mathematics in the popular press. So, don't read it--write a better one!

Lancy, David F. (Ed), *The indigenous mathematics project*, Papua New Guinea Journal of Education 14 (1978) 1-217.

The diversity in Papua New Guinea, where more than 700 distinct languages are spoken, offers special problems to its government in designing educational policies, programs, and materials. The papers collected here describe facets of an unusually ambitious project to research indigenous mathematics systems (e.g., counting systems), conduct cross-cultural cognitive testing, uncover variation in the quality of instruction in mathematics classrooms, and devise teaching strategies taking advantage of native mathematical strengths to promoting overall cognitive development.

Boulding, K., *Numbers count*, The Sciences 19:8 (October 1979) 79-82.

An economist on "epistemological statistics." The author is concerned with the impact of number and measuring systems on the way people view the world. In particular, he discussed advantages of number bases other than 10 and of measuring systems other than metric.

Frankel, Theodore, Gravitational Curvature: An Introduction to Einstein's Theory, Freeman, 1979; xviii + 172 pp, $18.50, $8.95 (P).

This book gives a rare almost-understandable glimpse at how the Einstein theory employs differential geometry. Good layout and short chapters aid readability, but a comfortable command of differential geometry is a sine qua non.

*NRC urges reform of undergraduate mathematics*, Science News 116 (1 Sept. 1979) 152.

A panel chaired by Peter Hilton stresses the need for more relevant mathematics in the undergraduate curriculum.

Amir-Moez, Ali R., String Figures: A Symbolic Approach, Western Printing (2902 Texas Ave., Lubbock, TX 79405), 1979; $3 (P) (English); $6 (P) (Farsee).

An exposition of the author's algebraic notation for string figures ("cat's cradle"), propounded earlier in the *Journal of Recreational Mathematics*.

McCorduck, P., Machines Who Think, Freeman, 1979, xiv + 375 pp, $14.95.

A history of artificial intelligence. The book is more a social history (with personal interviews) of the development of artificial intelligence as a field than a history of technical breakthroughs.

Chan, T.F. and Lewis, J.G., *Computing standard deviation: Accuracy*, Comm. Assoc. Comp. Mach. 22:9 (September 1979) 526-532.

To how many places should a standard deviation be calculated? The authors present a standard based on the accuracy of the data.

Floyd, Robert W., *The paradigms of programming*, Comm. Assoc. Comp. Mach. 22:8 (August 1979) 455-460.

The 1978 Turing Lecture, to the effect that "continued advances in programming will require the continuing invention, elaboration, and communication of new paradigms" (such as structured programming), together with the design of programming languages which will support those paradigms. In other words, new languages must reflect better the ways in which we think.

Kespohl, Ruth Carwell, Geometry Problems My Students Have Written, NCTM, 1979; vii + 87 pp, $5.80 (P).

A refreshing collection of constructions, computations on right triangles, and other problems too hard to classify, with solutions. Each problem is "clothed" in a creative tale or setting; textbook exercises should show so much imagination!

Høyrup, Else, Women and Mathematics, Science and Engineering: A Partially Annotated Bibliography with Emphasis on Mathematics and with References on Related Topics, Roskilde University Library, Denmark, 1978; vi + 62 pp, (P).

Collected in one place here are references to much of the literature in European languages on sex differences and sex roles in mathematics and science. Included is a skeletal list of biographical material on women mathematics.

Dauben, Joseph Warren, Georg Cantor: His Mathematics and Philosophy of the Infinite, Harvard U Pr, 1979; xiii + 404 pp, $27.50.

Thoroughly researched and engagingly written, this first complete scientific biography of Cantor should dispel once and for all the myths that have surrounded him (e.g., E.T. Bell's erroneous account in *Men of Mathematics*, which includes among other things the false claim that Cantor was Jewish.)

# NEWS & LETTERS

## MAGIC SQUARES

Here are general forms for 4 x 4 magic squares, with magic constant $N$ of the form $4n$, $4n+1$, $4n+2$, or $4n+3$:

| | | | |
|---|---|---|---|
| $n-8$ | $n+7$ | $n+6$ | $n-5$ |
| $n+4$ | $n-3$ | $n-2$ | $n+1$ |
| $n-1$ | $n+2$ | $n+3$ | $n-4$ |
| $n+5$ | $n-6$ | $n-7$ | $n+8$ |

$N = 4n$

| | | | |
|---|---|---|---|
| $n-8$ | $n+8$ | $n+6$ | $n-5$ |
| $n+4$ | $n-3$ | $n-1$ | $n+1$ |
| $n$ | $n+2$ | $n+3$ | $n-4$ |
| $n+5$ | $n-6$ | $n-7$ | $n+9$ |

$N = 4n+1$

| | | | |
|---|---|---|---|
| $n-7$ | $n+7$ | $n+6$ | $n-4$ |
| $n+4$ | $n-2$ | $n-1$ | $n+1$ |
| $n$ | $n+2$ | $n+3$ | $n-3$ |
| $n+5$ | $n-5$ | $n-6$ | $n+8$ |

$N = 4n+2$

| | | | |
|---|---|---|---|
| $n-8$ | $n+8$ | $n+7$ | $n-4$ |
| $n+5$ | $n-2$ | $n-1$ | $n+1$ |
| $n$ | $n+2$ | $n+4$ | $n-3$ |
| $n+6$ | $n-5$ | $n-7$ | $n+9$ |

$N = 4n+3$

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## NEW UMAP-MAA JOURNAL

The MAA has decided to cooperate with UMAP in the publication of a new quarterly journal devoted to the use of modules in undergraduate mathematics and its applications. This new journal, to be edited by Ross Finney and published by Birkhäuser Boston, will appear early in 1980. UMAP, the Undergraduate Mathematics and its Applications Project, is an NSF funded program devoted to the preparation and dissemination of supplementary modular classroom material in applications of undergraduate mathematics. The new journal will contain modules as well as articles about the use of modules in classroom instruction. Inquiries should be addressed to Ross Finney, Editor, UMAP, 55 Chapel Street, Newton, MA 02160.

## THE 1979 U.S.A. MATHEMATICAL OLYMPIAD

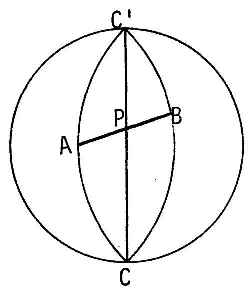
*The eighth U.S.A. Mathematical Olympiad took place on May 1, 1979, and the problems were published that same month in this column. The following sketches of solutions were adapted by Loren Larson from Samuel Greitzer's pamphlet "Olympiads for 1979." This pamphlet, containing more complete discussion of many of the problems, may be obtained for $.50 from Dr. Walter Mientka, Executive Director, MAA Committee on High School Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, NE 68588.*

1. Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \ldots + n_{14}^4 = 1{,}599 .$$

*Sol.* If $n_i$ is even then $n_i^4 \equiv 0$ (mod 16), and if $n_i$ is odd, $n_i^4 \equiv 1$ (mod 16). Thus $\sum n_i^4 \equiv 14$ (mod 16) at most, whereas $1599 \equiv 15$ (mod 16). Hence the Diophantine equation has no solutions.

2. A great circle $E$ on a sphere is one whose center is the center $O$ of the sphere. A pole $P$ of the great circle $E$ is a point on the sphere such that $OP$ is perpendicular to the plane of $E$. On any great circle through $P$, two points $A$ and $B$ are chosen equidistant from $P$. For any spherical triangle $ABC$ (the sides are great circle arcs) where $C$ is on $E$, prove that the great circle arc $CP$ is the angle bisector of angle $C$.




*Sol.* The great circle arcs $CA$, $CP$, $CB$ meet when extended at a point $C'$ diametrically opposite the point $C$. It is given that arc $PA$ = arc $PB$. Also arc $PC$ = arc $PC'$. Finally, spherical vertical angle $APC = BPC'$. Therefore spherical $\Delta APC \cong \Delta BPC'$. Similarly, $\Delta APC' \cong \Delta BPC$. The congruent triangles therefore have equal areas. On addition, we find that $\Delta APC + \Delta APC' = \Delta BPC + \Delta BPC'$. Therefore, lune $ACPC'$ = lune $BCPC'$ have equal areas. Since the area of a lune is measured by its spherical angle, $\angle PCA = \angle PCB$.

3. Given three identical n-faced dice whose corresponding faces are identically numbered with arbitrary integers. Prove that if they are tossed at random, the probability that the sum of the top three face numbers is divisible by three is greater or equal to 1/4.

*Sol.* We may replace the numbers on the faces of each die by their residues mod 3. Let $x, y, z$ be the probabilities, respectively, that the numbers 0,1,2 (mod 3) come up on each die. Then the desired probability is

$$p = x^3 + y^3 + z^3 + 6xyz \ .$$

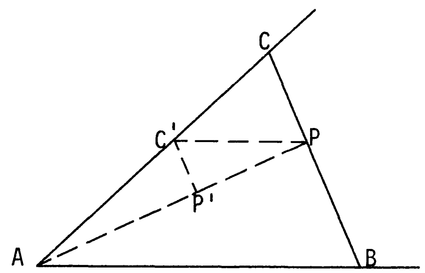
We may assume that $x \geq y \geq z$, so that $x \geq \frac{1}{3}$. One way of proceeding is to write $p$ in the following form (remembering that $x + y + z = 1$):

$$\begin{aligned} p &= x^3 + (y+z)^3 + 3yz(2x-y-z) \\ &= x^3 + (1-x)^3 + 3yz(3x-1) \\ &= 3x^2 - 3x + 1 + 3yz(3x-1) \\ &= 3(x-\tfrac{1}{2})^2 + \tfrac{1}{4} + 3yz(3x-1) \ . \end{aligned}$$

Since $(3x-1) \geq 0$ and $(x-\frac{1}{2})^2 \geq 0$, we have $p \geq \frac{1}{4}$.

4. Show how to construct a chord $BPC$ of a given angle $A$ through a given point $P$ within the angle $A$ such that $1/BP + 1/PC$ is a maximum.

*Sol.* Construct $PC'$ parallel to $AB$ and $C'P'$ parallel to $BC$.




From similar triangles $AP'C'$ and $APC$, $P'C' : PC = AP' : AP$. From similar triangles $PC'P'$ and $ABP$, we have $P'C' : BP = PP' : AP$. Adding the two proportions, we find

$$\frac{P'C'}{BP} + \frac{P'C'}{PC} = 1 \ ,$$

whence

$$\frac{1}{BP} + \frac{1}{PC} = \frac{1}{P'C'} \ .$$

To maximize the left side, make $P'C'$ perpendicular to $AP$; that is, $BC$ is to be constructed perpendicular to $AP$.

5. A certain organization has $n$ members $(n \geq 5)$ and it has $n + 1$ three member committees, no two of which have identical membership. Prove that there are two committees which share exactly one member.

*Sol.* Assume that given a pair of committees, they are either disjoint or have two common members. Since each of the $n + 1$ committees has three members and there are only $n$ persons involved, one person must belong to at least four committees. Let $A$ denote one such person and let $C_1$, $C_2$, $C_3$, $C_4$ denote four committees to which he belongs. If the membership of $C_1$ is $\{A,B,C\}$, then either $B$ or $C$ belongs to two of the remaining three committees $C_2$, $C_3$, $C_4$. Thus we may assume the following situation:

$C_1$: $A\ B\ C$

$C_2$: $A\ B\ D$

$C_3$: $A\ B\ E$

$C_4$: $A\ .\ .$

Now, $C_4$ must also contain $B$, for otherwise it cannot share two members with each of $C_1$, $C_2$ and $C_3$. The same is true for any additional committee to which $A$ belongs; if $A$ belongs to $C_1, C_2, \ldots, C_k$, then so does $B$. We thus account for $k$ committees with a total membership of $k + 2$. The remaining $\alpha = n + 1 - k$ committees are disjoint from $C_1, \ldots, C_k$ and involve $n - (k+2)$ $(= \alpha-3)$ members. If $\alpha \leq 7$, this is impossible. Otherwise, we may repeat the argument, finding a person who is a member of at least four committees, etc. Thus, finally, we obtain a contradiction.

## THE 1979 INTERNATIONAL MATHEMATICAL OLYMPIAD

*The September issue of Mathematics Magazine contained the problems from the 21st International Mathematical Olympiad, which took place in London in July 1979. Here are sketches of solutions to these olympian problems for readers who wish aid or confirmation. These too were adapted by Loren Larson from "Olympiads for 1979."*

1. Let $p$ and $q$ be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}\,.$$

Prove that $p$ is divisible by 1979.

*(W. Germany)*

*Sol.* We add and subtract terms as follows:

$$\begin{aligned}\frac{p}{q} &= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1319}\right) - 2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{1318}\right)\\ &= \left(1+\frac{1}{2}+\cdots+\frac{1}{1319}\right) - \left(1+\frac{1}{2}+\cdots+\frac{1}{659}\right)\\ &= \frac{1}{660} + \frac{1}{661} + \cdots + \frac{1}{1319}\\ &= \left(\frac{1}{660}+\frac{1}{1319}\right) + \left(\frac{1}{661}+\frac{1}{1318}\right) + \cdots + \left(\frac{1}{989}+\frac{1}{990}\right)\\ &= \frac{1979}{660\cdot 1319} + \frac{1979}{661\cdot 1318} + \cdots + \frac{1979}{989\cdot 990}\,.\end{aligned}$$

The result follows since 1979 is a prime number and the denominators on the right side are each relatively prime to 1979.

2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line-segments $A_iB_j$, for all $i, j = 1, \ldots, 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color.

*(Bulgaria)*

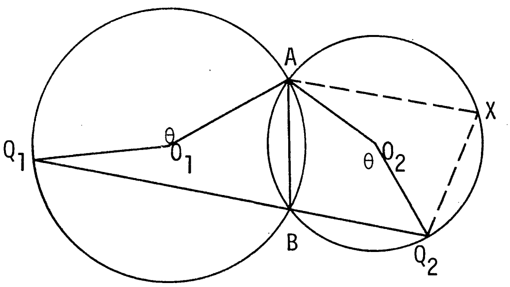
*Sol.* Suppose that edge $A_1A_2$ is, say, red and $A_2A_3$ is green. Among the lines $A_2B_1$, $A_2B_2$, $A_2B_3$, $A_2B_4$, $A_2B_5$, at least three have the same color. Suppose these are red, and label them $A_2B_i$, $A_2B_j$, $A_2B_k$. At least one of the segments $B_iB_j$, $B_jB_k$, $B_kB_i$ is an edge of the base; call it $B_rB_s$. If $B_rB_s$ were red we would have a "red" triangle $A_2B_rB_s$. On the other hand, if $B_rB_s$ were green, $A_1B_rB_s$ would be a green triangle (the lines $A_1B_r$ and $A_1B_s$ are green, for otherwise we would have red triangles $A_1A_2B_r$ or $A_1A_2B_s$). In the same manner, if three lines among $A_2B_1$, $A_2B_2$, $A_2B_3$, $A_2B_4$, $A_2B_5$ were green, we could find a monochromatic triangle (using the fact that $A_2A_3$ is green). This contradiction implies that $A_1A_2$ and $A_2A_3$ have the same color.

Similarly, $A_2A_3$ and $A_3A_4$ have the same color, and so on. We conclude that all the edges of the base $A_1A_2A_3A_4A_5$ have the same color. Likewise, the edges of the base $B_1B_2B_3B_4B_5$ have the same color.

Suppose that the edges of $A_1A_2A_3A_4A_5$ are red but that those of the base $B_1B_2B_3B_4B_5$ are green. Again, consider the segments $A_2B_1$, $A_2B_2$, $A_2B_3$, $A_2B_4$, $A_2B_5$. If three of these are red we get a monochromatic triangle using the argument in the first paragraph. However, if three of them are green, we then get a green triangle with vertex at $A_2$, again a contradiction. We must conclude that all ten edges of the two bases have the same color.

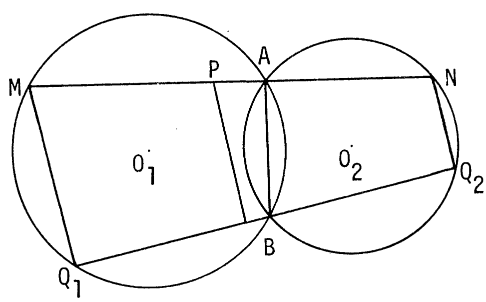
3. Two circles in a plane intersect. Let $A$ be one of the points of intersection. Starting simultaneously from $A$ two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to $A$ simultaneously after one revolution. Prove that there is a fixed point $P$ in the plane such that, at any time, the distances from $P$ to the moving points are equal. *(USSR)*

*Sol.* In the figure below, $Q_1$ on circle $O_1$ and $Q_2$ on circle $O_2$ have traveled counterclockwise through an equal angle $\theta$. Draw $Q_1B$ and $Q_2B$,




where $B$ is the other intersection of the two circles. Then $\angle ABQ_1 = \theta/2$ and $\angle ABQ_2 = \pi - \frac{\theta}{2}$, so that for all positions of $Q_1$ and $Q_2$, the line $Q_1Q_2$ always passes through the point $B$.

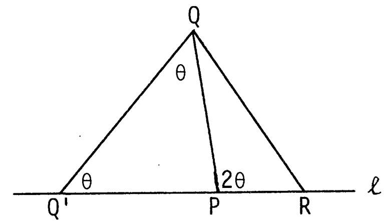
Now let segment $MN$ be the line segment passing through $A$ and perpendicular to $AB$ and let points $Q_1$ and $Q_2$ represent the positions of the two points at any arbitrary time (see figure).




Note that $\angle MQ_1B$ and $\angle BQ_2N$ are right angles. It follows that the perpendicular bisector of segment $Q_1Q_2$ will be parallel to $Q_1M$ and $Q_2N$, and will intersect $MN$ in the desired point $P$ (the midpoint of segment $MN$).

4. Given a plane $\pi$, a point $P$ in this plane and a point $Q$ not in $\pi$, find all points $R$ in $\pi$ such that the ratio $(QP + PR)/QR$ is a maximum. *(USA)*

*Sol.* First, suppose that $R$ lies on a line $\ell$ that passes through $P$. Let the angle between $PQ$ and $\ell$ be $2\theta$.




On line $\ell$, locate point $Q'$ such that $PQ' = PQ$. Then

$$\frac{QP+PR}{QR} = \frac{Q'R}{QR} = \frac{\sin Q'QP}{\sin\theta}.$$

Thus, on $\ell$, the maximum value is $\frac{1}{\sin\theta}$ and occurs when $\angle Q'QR$ is a right angle.

The minimum value for $\sin\theta$ occurs when $2\theta$ is a minimum, and this will occur when $\ell$ passes through $P$ and through the foot of the perpendicular form $Q$ to the plane $\pi$. Thus, if $QP$ is perpendicular to $\pi$, the points $R$ lie on a circle of radius $QP$ and center $P$, whereas, if $QP$ is not perpendicular to $\pi$, the maximum occurs at a unique point $R$.

5. Find all real numbers $a$ for which there exist non-negative real numbers $x_1, x_2, x_3, x_4, x_5$ satisfying the relations

$$\sum_{k=1}^{5} kx_k = a,\quad \sum_{k=1}^{5} k^3x_k = a^2,\quad \sum_{k=1}^{5} k^5x_k = a^3.$$ *(Israel)*

*Sol.* Multiply the equation by $a^2, -2a$, 1 respectively, and add to obtain $\sum a^2kx_k - 2\sum ak^3x_k + \sum k^5x_k = 0$, or equivalently $\sum x_kk(a-k^2)^2 = 0$. Hence either $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, or four of the $x_k$ are zero, while the fifth, say $x_j$, is such that $x_j = j$ and $a = j^2$. Therefore the possible values for $a$ will be $a = 0, 1, 4, 9, 16, 25$.

6. Let $A$ and $E$ be opposite vertices of a regular octagon. A frog starts

jumping at vertex $A$. From any vertex of the octagon except $E$, it may jump to either of the two adjacent vertices. When it reaches vertex $E$, the frog stops and stays there. Let $a_n$ be the number of distinct paths of exactly $n$ jumps ending at $E$. Prove that $a_{2n-1} = 0$,

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}),\quad n = 1,2,3,\ldots,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

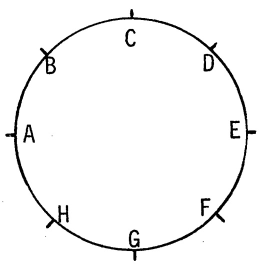
*(W. Germany)*

Note: A path of $n$ jumps is a sequence of vertices $(P_0,\ldots,P_n)$ such that

i) $P_0 = A$, $P_n = E$;

ii) for every $i$, $0 \le i \le n-1$, $P_i$ is distinct from $E$;

iii) for every $i$, $0 \le i \le n-1$, $P_i$ and $P_{i+1}$ are adjacent.




*Sol.* It is clear that $a_1 = a_2 = a_3 = 0$ and $a_4 = 2$. An odd number of jumps will get the frog to $B,D,F,H$ from $A$, but never to $E$, so $a_{2n-1} = 0$. Let $b_n$ be the number of distinct paths of exactly $n$ jumps ending at $C$ ($b_n$ is also the number ending at $G$). In getting to $E$, the frog must pass through $D$ or $F$. Hence

(1) $\quad a_n = 2a_{n-2} + 2b_{n-2}$

(2) $\quad b_n = 2b_{n-2} + a_{n-2}$

for $n > 2$.

Subtracting, $b_n - a_n = -a_{n-2}$, and changing subscripts,

(3) $\quad b_{n-2} = a_{n-2} - a_{n-4}$.

Substituting in (1) we obtain

(4) $\quad a_n = 4a_{n-2} - 2a_{n-4}$

for $n > 4$.

Instead of solving this recursive equation, it is simpler to check that the given expression satisfies the recursion (4). First, note that $x$ and $y$ are roots of $t^2 - 4t + 2 = 0$, and hence also of $t^{n-1} - 4t^{n-2} + 2t^{n-3} = 0$. Thus $a_{2n} - 4a_{2n-2} + 2a_{2n-4}$, which equals

$$\frac{1}{\sqrt{2}}(x^{n-1}-y^{n-1}) - \frac{4}{\sqrt{2}}(x^{n-2}-y^{n-2}) + \frac{2}{\sqrt{2}}(x^{n-3}-y^{n-3})$$

or equivalently,

$$\frac{1}{\sqrt{2}}(x^{n-1}-4x^{n-2}+2x^{n-3}) - \frac{1}{\sqrt{2}}(y^{n-1}-4y^{n-2}+2y^{n-3})$$

must be zero.

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